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CONTENTS

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On the frequency of Titchmarsh's phenomenon for $\zeta(s)$—VIII

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Abstract. For suitable functions $H = H(T)$ the maximum of $|(\zeta(\sigma + it))^z|$ taken over $T \leqslant t \leqslant T + H$ is studied. For fixed $\sigma(\frac{1}{2} \leqslant \sigma \leqslant 1)$ and fixed complex constants z "expected lower bounds" for the maximum are established.

Keywords. Riemann zeta-function; frequency; Titchmarsh's phenomenon.

1. Introduction

It is our object to prove the following three Ω-theorems by applying two fundamental theorems of Ramachandra which he proved in [5] (these theorems will be stated in § 2). Let $G(\sigma, t) = |(\zeta(\sigma + it))^z|$ where σ is a constant in $[\frac{1}{2}, 1]$ and z is a non-zero complex constant. Since z can be written as $z = re^{i\theta}$ where $r > 0$ and θ lies in $[0, 2\pi)$, in order to state Ω theorems for $G(\sigma, t)$ it suffices to assume $r = 1$. *We shall in what follows write* $z = e^{i\theta}$.

Theorem 1. *We have, with* $z = e^{i\theta}$,

$$\max_{T \leqslant t \leqslant T+H} G(1, t) \geqslant e^{\gamma} \lambda(\theta)(\log\log H - \log\log H) + O(1)$$

where γ is the Euler's constant and

$$\lambda(\theta) = \prod_p \left\{ \left(1 - \frac{1}{p}\right) \left(\frac{p[(p^2 - \sin^2\theta)^{\frac{1}{2}} + \cos\theta]}{p^2 - 1} \right)^{\cos\theta} \times \exp\left(\sin\theta \tan^{-1}\left(\frac{\sin\theta}{(p^2 - \sin^2\theta)^{\frac{1}{2}}} \right) \right) \right\}.$$

The conditions on H and T are $T \geqslant H \geqslant C \log\log\log T$, $T \geqslant T_0$ where C and T_0 are large positive constants.

Remark 1. Levinson [3] was the first to prove that when $\theta = 0$

$$\max_{1 \leqslant t \leqslant T} G(1, t) \geqslant e^{\gamma} \log\log T + O(1)$$

and when $\theta = \pi$

$$\max_{1 \leqslant t \leqslant T} G(1,t) \geqslant \frac{6}{\pi^2} e^{\gamma} (\log\log T - \log\log\log T) + O(1).$$

Note that $\lambda(\pi) = 6/\pi^2$. However, around the same time Ramachandra [6] proved that

$$\max_{T \leqslant t \leqslant T+H} G(1,t) \geqslant e^{\gamma} \lambda(\theta) \log\log H (1 + O(1))$$

when $\theta = 0$, and $\theta = \pi$. Later Ramachandra [7] extended the conditions on H to $H \geqslant C \log\log\log\log T$ (without assuming any hypothesis) and to $H \geqslant C \log\log\log\log\log T$ (assuming Riemann hypothesis). These results go through for any θ in $[0, 2\pi)$.

Remark 2. This theorem as well as Theorems 2 and 3 has obvious extensions to ordinary L-functions and more generally to L-functions of algebraic number fields and so on. We do not carry out the details here.

Theorem 2. *Let α be a constant in $(\frac{1}{2}, 1)$. Then*

$$\max_{T \leqslant t \leqslant T+H} G(\alpha, t) > \exp\left(C_1 \frac{(\log H)^{1-\alpha}}{\log\log H} \right)$$

where C_1 is a positive constant depending on α and $T^{1/3} \leqslant H \leqslant T$.

The next theorem depends on Riemann hypothesis (R.H.).

Theorem 3. *(on R.H.). In Theorem 2 the condition on H can be relaxed to $T \geqslant H \geqslant C \log\log T$. Also under this condition on H, there holds*

$$\max_{T \leqslant t \leqslant T+H} G(\tfrac{1}{2}, t) > \exp\left(C_2 \left(\frac{\log H}{\log\log H} \right)^{\frac{1}{2}} \right)$$

where $C_2 > 0$ is a numerical constant.

Remark 1. When $\theta = 0$, Theorem 3 can be upheld without assuming R.H.

Remark 2. The results of this paper are inspired by the paper [4] of Montgomery who proves

$$\max_{0 \leqslant t \leqslant T} G(\alpha, t) > \exp\left(C_3 \frac{(\log T)^{1-\alpha}}{(\log\log T)^{\alpha}} \right)$$

where $C_3 \geqslant 0$ depends on α and $\frac{1}{2} < \alpha < 1$. In a recent paper [8], Ramachandra and Sankaranarayanan have obtained this result with $C_3 = C_4(\alpha - \frac{1}{2})^{\frac{1}{2}}/(1-\alpha)$ where $C_4 > 0$ is a numerical constant. (The quantity $(\alpha - \frac{1}{2})^{\frac{1}{2}}$ can be replaced by 1 if we assume R.H.). However Montgomery's method does not work for short intervals $[T, T+H]$ and also for L-functions.

2. Ramachandra's theorems

We shall now state a special case of the main theorem of [5].

Theorem 4. *Let $a_1 = 1, a_2, a_3, \ldots$ be a sequence of complex numbers satisfying $|a_n| \leqslant (nH)^A$ where $H \geqslant 10^{10}$ and A is a positive constant. Let $F(s) = \Sigma_{n=1}^{\infty} a_n/n^s$ (where $s = \sigma + it$) admit an analytic continuation in $\sigma > 0$, $T \leqslant t \leqslant T + H$. Then*

$$\max_{\sigma > 0}\left(\frac{1}{H}\int_T^{T+H} |F(\sigma + it)|^2 \,\mathrm{d}t\right) > C(A) \sum_{n \leqslant H/200} |a_n|^2 \left(1 - \frac{\log n}{\log H} + \frac{1}{\log\log H}\right),$$

where $C(A)$ is a positive constant depending only on A.

As a corollary he deduced

Theorem 5. *In addition to the condition of Theorem 4 let us suppose that in $(\sigma > 0, T \leqslant t \leqslant T + H)$ the maximum of $|F(s)|$ be $\leqslant \exp\exp(H/100A)$. Then*

$$\left(\frac{1}{H}\int_T^{T+H} |F(it)|^2 \,\mathrm{d}t\right) \geqslant C(A) \sum_{n \leqslant H/200} |a_n|^2 \left(1 - \frac{\log n}{\log H} + \frac{1}{\log\log H}\right)$$

where $C(A) > 0$ depends only on A, provided the LHS is interpreted in a limiting sense.

Remark 1. In the reference to Ramachandra's paper the theorem proved is slightly different. But it is not hard (from his argument) to prove Theorem 4 and and deduce Theorem 5. In that reference Ramachandra uses the kernel related to $\exp(s^{4a+2})$ where a is a non negative integer. However in deducing Theorem 5 from Theorem 4 we have to use the kernel related to $\exp((\sin s)^2)$.

Remark 2. From Theorem 4 we can also deduce that the maximum of $G(\sigma, t)$ in $(\sigma \geqslant 1, T \leqslant t \leqslant T + H)$ exceeds the right hand side of Theorem 1. Similar remarks holds good for Theorems 2 and 3.

Remark 3. For improvements of Theorems 4 and 5 see the paper [2] by Balasubramanian and Ramachandra.

3. Proof of theorem 1.

We will prove Theorems 1, 2 and 3 in §§ 3, 4 and 5 respectively. We adopt different notations in each of these sections and we will explain the notations in each of these sections in the respective sections.

Lemma 3.1. *Let k be a positive integer and let*

$$F(s) = (\zeta(1 + s))^{kz} = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{3.1}$$

then

$$a_1 = 1 \text{ and } |a_n| \leqslant n^2(\zeta(2))^k. \tag{3.2}$$

Proof. Follows by Euler product for $\zeta(1+s)$.

Lemma 3.2. *We have in* $(\sigma \geqslant 0, T \leqslant t \leqslant T+H)$

$$|F(s)| \leqslant \exp(k C_1 \log\log T) \tag{3.3}$$

where c_1 *is a positive constant and* $10 < H \leqslant T$.

Proof. Follows since it is well-known that

$$\log \zeta(\sigma + it) = O(\log\log T)$$

in $(\sigma \geqslant 1, T \leqslant t \leqslant T+H)$.

Lemma 3.3. *The conditions for the application of Theorem* 5 *are satisfied if* $H \gg \log\log\log T$ *where the implied constant is a large positive constant, and* $k = O(\log H)$.

Proof. Follows from Lemmas 3.1 and 3.2.

Lemma 3.4. *Let* $k_0 = kz$ *and let* $n \geqslant 2$ *and let* $n = \prod_p p^m$ *be the prime power decomposition of* n. *Then*

$$a_1 = 1 \text{ and } a_n = \prod_p \frac{k_0(k_0+1)\ldots(k_0+m-1)}{m!\,p^m} \tag{3.4}$$

Proof. Follows by Euler's product for $\zeta(1+s)$.

Lemma 3.5. *Let, for each* $p \leqslant k$,

$$l = k\left(\frac{\cos\theta + (p^2 - \sin^2\theta)^{\frac{1}{2}}}{p^2 - 1}\right) = \frac{k}{q} \text{ say and } m = [l]. \tag{3.5}$$

Then, putting

$$n = \prod_{p \leqslant k} p^m, \text{ we have}$$

$$\frac{1}{2k}\log|a_n|^2 = \frac{1}{2k}\sum_{p \leqslant k}\{-2m\log m + 2m + O(\log m) - 2m\log p + E(k,m)\} \tag{3.6}$$

where

$$E(k,m) = \sum_{v=0}^{m-1} \log(k^2 + v^2 + 2kv\cos\theta). \tag{3.7}$$

Proof. Follows from the formula

$$\log m! = m\log m - m + O(\log m).$$

Lemma 3.6. *We have,*

$$E(k,m) = 2m\log k + k\int_0^{1/q} \log(1 + u^2 + 2u\cos\theta)\,du + O\left(\frac{1}{p}\right). \tag{3.8}$$

Proof. We have

$$E(k,m)=\sum_{v=0}^{m-1}\left\{\log(k^2+v^2+2kv\cos\theta)-\int_v^{v+1}\log(k^2+u^2+2ku\cos\theta)\mathrm{d}u\right\}+\int_0^m\log(k^2+u^2+2ku\cos\theta)\,\mathrm{d}u.$$

Here the sum on the right is easily seen to be $O(1/p)$. The integral on the right is

$$2m\log k+\int_0^m\log\left(1+\frac{u^2}{k^2}+2\frac{u}{k}\cos\theta\right)\mathrm{d}u.$$

Here we can replace the upper limit m of the integral by l with an error $O(m/k)=O(1/p)$. The lemma now follows by a change of variable.

Lemma 3.7. *We have,*

$$\frac{1}{k}\sum_{p\leqslant k}\log m=O\left(\frac{1}{\log k}\right)\tag{3.9}$$

and

$$\frac{1}{k}\sum_{p\leqslant k}\frac{1}{p}=O\left(\frac{1}{\log k}\right).$$

Proof. Follows by prime number theorem.

Lemma 3.8. *We have,*

$$\frac{1}{2k}\sum_{p\leqslant k}\{-2m\log m+2m-2m\log p+2m\log k\}=\sum_{p\leqslant k}\left\{-\frac{1}{q}\log\frac{p}{q}+\frac{1}{q}\right\}+O\left(\frac{1}{\log k}\right).\tag{3.10}$$

Proof. On the LHS we can replace m by l with a total error

$$\leqslant\frac{1}{2k}\sum_{p\leqslant k}O(\log m)=O\left(\frac{1}{\log k}\right).$$

The rest is

$$\sum_{p\leqslant k}\left\{-\frac{1}{q}\log\frac{k}{q}+\frac{1}{q}-\frac{1}{q}\log p+\frac{1}{q}\log k\right\}$$

which gives the lemma.

Lemma 3.9. *We have,*

$$\frac{1}{2k}\sum_{p\leqslant k} k\int_0^{(1/q)} \log(1+u^2+2u\cos\theta)\mathrm{d}u$$
$$=\operatorname{Re}\sum_{p\leqslant k}\left(\frac{1+(1/q)\mathrm{e}^{i\theta}}{\mathrm{e}^{i\theta}}\log\left(1+\frac{1}{q}\mathrm{e}^{i\theta}\right)-\frac{1}{q}\right). \tag{3.11}$$

Proof. Trivial.

Lemma 3.10. *We have,*

$$\frac{1}{2k}\log|a_n|^2=\log\log k+\gamma+\log\lambda(\theta)+O\left(\frac{1}{\log k}\right), \tag{3.12}$$

where $\lambda(\theta)$ *is as in Theorem* 1.

Proof. By Lemmas 3.5, 3.6, 3.7 and 3.8 we see that LHS of (3.12) is, (with an error $O(1/\log k)$),

$$\operatorname{Re}\sum_{p\leqslant k}\left\{-\frac{1}{q}\log\frac{p}{q}+\frac{1}{q}\log\left(1+\frac{1}{q}\mathrm{e}^{i\theta}\right)+\mathrm{e}^{-i\theta}\log\left(1+\frac{1}{q}\mathrm{e}^{i\theta}\right)\right\}.$$

Now the contribution from the first two terms (in the curly bracket) to the sum is

$$\operatorname{Re}\sum_{p\leqslant k}\frac{1}{q}\log\left|\frac{q+\mathrm{e}^{i\theta}}{p}\right|=0,$$

since

$$\begin{aligned}|q+\mathrm{e}^{i\theta}|^2&=\left(\frac{p^2-1}{(p^2-\sin^2\theta)^{\frac12}+\cos\theta}+\cos\theta\right)^2+\sin^2\theta\\&=\left(\frac{p^2-1+\cos^2\theta+\cos\theta(p^2-\sin^2\theta)^{\frac12}}{(p^2-\sin^2\theta)^{\frac12}+\cos\theta}\right)^2+\sin^2\theta\\&=p^2.\end{aligned}$$

The third term contributes

$$\begin{aligned}&\sum_{p\leqslant k}\left(\cos\theta\log\frac{p}{q}+\sin\theta\tan^{-1}\left(\frac{\sin\theta}{q+\cos\theta}\right)\right)\\&\quad=\sum_{p\leqslant k}\left\{\log\left(1-\frac{1}{p}\right)+\cos\theta\log\frac{p}{q}+\sin\theta\tan^{-1}\left(\frac{\sin\theta}{q+\cos\theta}\right)\right\}\\&\qquad+\sum_{p\leqslant k}\log\left(1-\frac{1}{p}\right)^{-1}.\end{aligned}$$

This together with the well-known formula $\Pi_{p\leqslant k}(1-1/p)^{-1}=\mathrm{e}^{\gamma}\log k+O(1)$ proves the lemma.

Lemma 3.11. *For the n defined in Lemma 3.5, we have,*

$$\log n = \sum_{p \leqslant k} m \log p = k \log k + O(k). \tag{3.13}$$

Proof. Replacement of m by l involves an error $O(k)$ by the prime number theorem. Now $l = k/q$ and

$$\begin{aligned} q &= p\left(p - \frac{1}{\mathrm{p}}\right)(p[1-(\sin^2\theta/p^2)]^{\frac{1}{2}} + \cos\theta)^{-1} \\ &= p\left(1 - \frac{1}{p^2}\right)\left([1-(\sin^2\theta/p^2)]^{\frac{1}{2}} + \frac{\cos\theta}{p}\right)^{-1} \\ &= p + O(1). \end{aligned}$$

This proves the lemma.

Lemma 3.12. *Set* $k = [\log H/(2\log\log H)]$. *Then for all H exceeding a large positive constant, we have,*

$$n \leqslant \frac{H}{200}.$$

Proof. Follows from Lemma 3.11.

Lemma 3.13. *The maximum of* $|(\zeta(1+s))^z|$ *in* $(\sigma = 0,\ T \leqslant t \leqslant T+H)$ *exceeds*

$$\left(\frac{C(A)}{\log\log H}|a_n|^2\right)^{1/2k}.$$

Proof. Follows from Theorem 5.

Lemmas 3.12 and 3.13 complete the proof of Theorem 1, in view of Lemma 3.10.

4. Proof of theorem 2.

Lemma 4.1. *Let* $\frac{1}{2} \leqslant \beta \leqslant 1$ *and* $H = T^{1/3}$. *Then the number of zeros of* $\zeta(s)$ *in* $(\sigma \geqslant \beta,\ T \leqslant t \leqslant T+H)$ *is*

$$\ll H^{4(1-\beta)/(3-2\beta)}(\log T)^{100} \tag{4.1}$$

where the constant implied by the Vinogradov symbol $\ll$ *is absolute.*

Proof. This is a consequence of a deep result of Balasubramanian [1] on the mean-square of $|\zeta(\frac{1}{2}+it)|$. (See Theorem 6 on page 576 of his paper).

Lemma 4.2. *Let* $\frac{1}{2} < \beta < \alpha < 1$. *Then there exists a t-interval* I *contained in* $T \leqslant t \leqslant T+H$, *of length* T^δ *(where* $\delta > 0$ *is a constant depending on* β*) such that the region* $(\sigma \geqslant \beta,\ t \in I)$ *is free from zeros of* $\zeta(s)$.

Proof. Follows from Lemma 4.1.

Lemma 4.3. *Let I_0 denote the t-interval obtained from I by removing on both sides intervals of length* (1/100) T^{δ}. *Then in* $(\sigma \geqslant \alpha, t \in I_0)$ *we have*

$$\log \zeta(s) = O(\log T).$$

Proof. Follows by Borel-Caratheodory theorem.

Lemma 4.4. *We apply Theorem 5 to the interval I_0 in place of $T \leqslant t \leqslant T + H$. Then*

$$\max_{(\sigma = 0, t \in I_0)} |(\zeta(\alpha + s))^z| \geqslant \left(\frac{C(A)}{\log \log H} |a_n|^2 \right)^{1/2k} \tag{4.2}$$

where $n \leqslant H/200$, $k \geqslant 1$ is any integer which is $O(\log H)$, and a_n are defined by

$$F(s) = (\zeta(\alpha + s))^{kz} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \tag{4.3}$$

Proof. It is easily seen (as before) that the conditions for the application of Theorem 5 are satisfied and hence the lemma.

Lemma 4.5. *Let*

$$n = \prod_{(k/4)^{1/\alpha} \leqslant p \leqslant (k/2)^{1/\alpha}} p. \tag{4.4}$$

Then

$$|a_n|^2 \geqslant \exp\left(\frac{C_1 k^{1/\alpha}}{\log k} \right).$$

where C_1 is a positive constant.

Proof. Follows by Euler's product for $\zeta(s)$.

Lemma 4.6. *Let $k = [C_2 (\log H)^{\alpha}]$ where $C_2 > 0$ is a small constant. Then, we have,*

$$n \leqslant \frac{H}{200} \tag{4.5}$$

and so R.H.S. *of* (4.2) *exceeds*

$$\exp\left(\frac{C_3 (\log H)^{1-\alpha}}{\log \log H} \right), \tag{4.6}$$

where $C_3 > 0$ is a constant.

Proof. Follows from lemma 4.5.

Theorem 2 now follows from (4.2) and lemma 4.6.

5. Proof of theorem 3

The first part of Theorem 3 follows exactly as in the proof of Theorem 2. It remains to prove only the second part of Theorem 3. We begin with

Lemma 5.1. *Given any $t \geqslant 10$ there exists a real number τ with $|t-\tau| \leqslant 1$ such that*

$$\frac{\zeta'(\sigma+i\tau)}{\zeta(\sigma+i\tau)} = O((\log t)^2) \tag{5.1}$$

uniformly in $-1 \leqslant \sigma \leqslant 2$. Hence

$$\log \zeta(\sigma+i\tau) = O((\log t)^2) \tag{5.2}$$

uniformly in $-1 \leqslant \sigma \leqslant 2$.

Proof. See Theorem 9.6 (A), p. 184 of [9].

Lemma 5.2. *Let*

$$F(s) = ((\zeta(\tfrac{1}{2}+s))^{kz}) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} (\sigma \geqslant 2), \tag{5.3}$$

where $k \geqslant 1$ is any integer. Let $x \geqslant 1000$,

$$\prod_{k^3 \leqslant p \leqslant k^4} \left(1 + \frac{k^2}{p^s}\right) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}. \tag{5.4}$$

Then

$$\sum_{n \leqslant x} \frac{b_n}{n} \leqslant \sum_{n \leqslant x} |a_n|^2, \tag{5.5}$$

$$\prod_{k^3 \leqslant p \leqslant k^4} \left(1 + \frac{k^2}{p}\right) > \exp(k^2 \log 5/4) \tag{5.6}$$

$$\prod_{k^3 \leqslant p \leqslant k^4} \left(1 + \frac{k^2}{p^{1+\delta}}\right) < \exp(k^2 e^{-C_1}) \tag{5.7}$$

where $\delta = C_1/\log k$ and C_1 is any positive constant.

$$\sum_{n \leqslant x} \frac{b_n}{n} > \exp(k^2 \log 5/4) - x^{\delta} \sum_{n > x} \frac{b_n}{n^{1+\delta}} \tag{5.8}$$

$$> \frac{1}{2} \exp(k^2 \log 5/4) \tag{5.9}$$

provided

$$k \geqslant C_2 \left(\frac{\log x}{\log \log x}\right)^{\frac{1}{2}} \tag{5.10}$$

where C_2 is a positive constant.

Proof. Equation (5.5) is trivial. Equations (5.6) and (5.7) follow from $\log(1+y)<y$ and $>y-y^2/2$ for $0<y<1$. Equation (5.8) is trivial where (5.9) follows if $x^\delta \leqslant \exp(k^2 e^{-C_1})$ and C_1 is large. This leads to the condition (5.10) for the validity of (5.9).

Lemma 5.3. *We have, with* $k=[C_3(\log H/\log\log H)^{\frac{1}{2}}]$,

$$\left(\frac{C(A)}{\log\log H}\sum_{n\leqslant H/200}|a_n|^2\right)^{1/2k} > \exp\left(k\log\frac{100}{99}\right) \tag{5.11}$$

where C_3 *is a certain positive constant.*

Proof. Follows from (5.5), (5.9) and (5.10).

Lemma 5.4. *The condition* $|a_n|\leqslant(nH)^A$ *is satisfied for some* $A>0$.

Proof. Follows from the Euler product for $\zeta(s)$.

Lemma 5.5. *Without loss of generality we can assume that*

$$\max_{T\leqslant t\leqslant T+H}|F(it)|\leqslant\exp(\log H)^3). \tag{5.12}$$

Proof. Otherwise the required result follows.

Lemma 5.6. *The inequality* (5.12) *implies* (*subject to* $H\geqslant C_4\log\log T$ *where* $C_4>0$ *is a large constant*) *that*

$$\max_{\sigma\geqslant 0,\,T+(H/9)\leqslant t\leqslant T+(8H/9)}|F(\sigma+it)|\leqslant\exp((\log H)^4). \tag{5.13}$$

Proof. Let $s_0=\sigma_0+it_0$ where $0<\sigma_0\leqslant 1$ and $T+\frac{H}{9}\leqslant t_0\leqslant T+\frac{8H}{9}$. Consider the analytic function

$$\phi(s)=F(s)\exp\left(\left(\sin\left(\frac{s-s_0}{100}\right)\right)^2\right).$$

For any real $t\geqslant 10$ let t^* denote the real number τ given by Lemma 5.1.

Let R denote the rectangle with the following corners,

$$s_1=i\left(t_0-\frac{H}{10}\right)^*,\ s_2=i\left(t_0+\frac{H}{10}\right)^*$$

$$s_3=2+i\left(t_0+\frac{H}{10}\right)^* \text{ and } s_4=2+i\left(t_0-\frac{H}{10}\right)^*.$$

On the horizontal sides of R we have

$$|F(s)|=\exp(O(k(\log T)^2)).$$

On the vertical sides we have, by lemma 5.5,

$$|F(s)| \leqslant \exp((\log H)^3).$$

Lemma 5.6 now follows since (by maximum modulus principle) $|F(s_0)| = |\phi(s_0)| \leqslant$ maximum of $|\phi(s)|$ on the boundary of R provided $C_4 > 0$ is a large constant.

Lemma 5.7. The conditions for applying Theorem 5 are satisfied for the interval $T + \frac{H}{9} \leqslant t \leqslant T + \frac{8H}{9}$.

Proof. Follows from Lemmas 5.4 and 5.6.

The second part of Theorem 3 now follows from Lemma 5.3 (by a slight change of notation).

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1)* R Balasubramanian, *On the frequency of Titchmarsh's phenomenon for $\zeta(s)$* -IV, *Hardy-Ramanujan J*, **9** (1986), 1–10

2)* K Ramachandra, *On the frequency of Titchmarsh's phenomenon for $\zeta(s)$*-IX, *Hardy-Ramanujan J*, **13** (1990), 28–33

in [1]* the author has shown that we can take $C_2 = 3/4$.

in [2]* the author has shown that if

$$\min_{|I|=H} \max_{t\in I} G(1,t) = f(H)$$

the minimum being over all intervals I of length H, then, for all $H \geqslant H_0(\theta)$,

$$|f(H)e^{-\gamma}(\lambda(\theta))^{-1} - \log\log H| \leqslant \log\log\log H + O(1)$$

where

$$\lambda(\theta) = \prod_p \lambda_p(\theta),$$

$$\lambda_p(\theta) = \left(1 - \frac{1}{p}\right)\left(\left(1 - \frac{s^2}{p^2}\right)^{1/2} - \frac{c}{p}\right)^{-c} \exp\left(s \sin^{-1}\frac{s}{p}\right)$$

c and s being defined by $c + is = \exp(i\theta)$. It is not hard to see that $\lambda(\theta)$ is the same as before.

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Generalized parabolic sheaves on an integral projective curve

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Abstract. We extend the notion of a parabolic vector bundle on a smooth curve to define generalized parabolic sheaves (GPS) on any integral projective curve X. We construct the moduli spaces $M(X)$ of GPS of certain type on X. If X is obtained by blowing up finitely many nodes in Y then we show that there is a surjective birational morphism from $M(X)$ to $M(Y)$. In particular, we get partial desingularisations of the moduli of torsion-free sheaves on a nodal curve Y.

Keywords. Generalized parabolic sheaf; projective curve.

1. Introduction

In [1] we defined and studied GPBs (generalized parabolic bundles) on an irreducible nonsingular projective curve. The notion easily generalizes to a GPS (= generalized parabolic sheaf) on an integral projective curve X. A GPS is a torsion-free sheaf E together with an additional structure called parabolic structure over disjoint effective Cartier divisors $\{D_j\}_{j\in J}$, J a finite set (see Definitions 1.3, 1.4). In [1] we constructed moduli spaces for GPBs with parabolic structure of certain type over a single divisor (i.e. $J=$ singleton). Here we consider many divisors. Moreover, X being singular, the method used in [1] fails. Therefore we generalize the method of Simpson [4] for the construction of moduli spaces.

Theorem 1. *There exists a (coarse) moduli space $M_{X,J}(k,d)$ of semistable GPS F of rank k, degree d with parabolic structure over D_j given by a flag $\mathscr{F}^i: H^\circ(F\otimes\mathcal{O}_{D_j})\supset F_1^j(F)\supset 0, \forall_j\in J$ and weights $(0,\alpha)$, where $a_j=\dim F_1^j(F)$ and rational number α are fixed with $0<\alpha<1$. $M_{X,J}=M_{X,J}(k,d)$ is a projective variety of dimension $k^2(g-1)+1+\Sigma_j a_j(k \text{ degree } D_j-a_j)$, $g=$ arithmetic genus of X.*

If X is nonsingular, then $M(k,d)$ is normal. If further $(k,d)=1$, $a_j=$ multiple of k and α is close to 1 then $M(k,d)$ is nonsingular and is a fine moduli space.

Theorem 2. *Let X be the curve (proper transform) obtained by blowing up nodes $\{y_j\}_{j\in J}$ of an integral projective curve Y, $\pi_{XY}: X\to Y$ surjection. For $j\in J$, let $D_j=\pi_{XY}^{-1}(y_j)$, $a_j=k$. Then there exists a surjective birational morphism $f_{XY}: M_{X,J\cup J'}\to M_{Y,J'}$. In particular, if $J'=\phi$, $X=$ desingularization of Y, $(k,d)=1$, α close to 1, then $M_{X,J}$ is the desingularization of the moduli space $M_{Y,\phi}$ of semistable torsion-free sheaves on Y. Further, if X' is a partial desingularization of Y, obtained by blowing up $y_j, j\in J'$, $\pi_{X,X'}$:*

$X \to X', \pi_{X'Y}: X' \to Y$, then (with suitable D_j and parabolic structure as above) $f_{XY} = f_{X'Y} \circ f_{XX'}$. Thus $M_{X'J'}$ is a partial desingularization of $M_{Y,\phi}$.

There is a close relationship between torsion-free sheaves on a singular curve Y and GPS on its desingularization. An analogue of Theorem 2 holds if $\{y_j\}$ are ordinary cusps, and hopefully also in case each y_i is an ordinary n-tuple point with linearly independent tangents.

1. Preliminaries

Let X be an integral projective curve defined over an algebraically closed field k. Let ω_X denote the dualising sheaf on X, it is a torsion-free sheaf. For a torsion-free sheaf E on X we denote by $r(E)$ and $d(E)$ respectively the rank and degree of E. Let $\{D_j\} j \in J$ be finitely many effective divisors on X such that supports of D_j are mutually disjoint.

DEFINITION 1.1

A quasi-parabolic structure on E over D_j is a flag $\mathscr{F}^j(E)$ of vector subspaces of $H^0(E \otimes \mathcal{O}_{D_j})$ viz.

$$\mathscr{F}^j(E): F_o^j(E) \equiv H^0(E \otimes \mathcal{O}_{D_j}) \supset F_1^j(E) \supset \cdots \supset F_r^j(E) = 0.$$

DEFINITION 1.2

Let $\mathscr{F}(E) = \{\mathscr{F}^j(E)\}_{j \in J}$. A QPS is a pair $(E, \mathscr{F}(E))$ where E is a torsion-free sheaf and $\mathscr{F}(E)$ is a quasiparabolic structure on $\{D_j\} j \in J$ as above.

DEFINITION 1.3

A parabolic structure on E over D_j is a quasiparabolic structure $\mathscr{F}^j(E)$ (See. 1.1) together with an r_j-tuple of real numbers $\boldsymbol{\alpha}^j = (\alpha_1^j(E), \cdots, \alpha_{r_j}^j(E)), 0 \leqslant \alpha_1^j(E) < \cdots < \alpha_{r_j}^j \cdot (E) < 1$, called weights associated to $\mathscr{F}^j(E)$.

Let $m_i^j = \dim F_{i-1}^j(E) - \dim F_i^j(E), i = 1, \cdots, r_j$. Define $wt_j(E) = \Sigma_{i=1}^{r_j} m_i^j \alpha_i^j(E), wt E = \Sigma_j wt_j(E)$. Let par $d(E) = d(E) + wt(E)$, par $\mu(E) = \text{par}\, d(E)/r(E)$.

DEFINITION 1.4

A GPS (generalized parabolic sheaf) is a triple $(E, \mathscr{F}(E), \boldsymbol{\alpha})$ with $\mathscr{F}, \boldsymbol{\alpha}$ as in 1.1 and 1.3.

1.5

Let K be a subsheaf of E such that the quotient E/K is torsion-free in a neighbourhood of D. Let $h: K \to E$ be the inclusion map. Since D is a divisor and E/K is torsion-free, one has $\mathrm{Tor}_1^{\mathcal{O}}(E/K, \mathcal{O}_D) = 0$ and therefore $h|_D: K|_D \to E|_D$ is an injection. Hence $H^0(K \otimes \mathcal{O}_D)$ can be identified with a subspace $F_o^j(K)$ of $F_o^j(E)$. Define $F_i^j(K) = F_o^j(K) \cap F_i^j(E)$. This gives (after omitting repetitions) a flag $\mathscr{F}^j(K), j \in J$. The set $\{\alpha_l^j(K)\}$ of weights for K is a subset of $\{\alpha_i^j(K)\}$ defined as follows. One has $F_l^j(K) =$

$F_i^j(E) \cap F_o^j(K)$ for some i, let i_o be largest such i. Then $\alpha_i^j(K) := \alpha_{i_o}^j(E)$. Thus a subsheaf of a GPS with torsion-free quotient gets a natural structure of a GPS.

DEFINITION 1.6

A GPS$(E, \mathscr{F}(E), \boldsymbol{\alpha})$ is semistable (respectively stable) if for every (resp. proper) subsheaf K of E with torsion-free quotient, one has par $\mu(K) \leqslant$ (resp. $<$) par $\mu(E)$.

Remarks 1.7. (1) If E/K is not torsion-free, then we may still define $F_o^i(K) =$ image of $H^0(K \otimes \mathcal{O}_D)$ under $H^0(h|_D)$ and define $\mathscr{F}^j(K)$ by intersecting $F_o^j(K)$ with the flag $\mathscr{F}^j(E)$. Thus we can talk of wtK. If M is the largest subsheaf of E containing K, with E/M torsion-free and $r(K) = r(M)$ then par $\mu(K) \leqslant$ par $\mu(M)$. Then the condition of 1.6 is satisfied for every subsheaf K of E if $(E, \mathscr{F}(E), \boldsymbol{\alpha})$ is a semistable (resp. stable) GPS. (2) There exists a natural parabolic structure on a quotient sheaf also. Semistability and stability can also be defined equivalently using quotients instead of subsheaves. (See 3.4, 3.5 [1]).

Assumptions 1.8. In this paper we want to study moduli spaces of GPS$(F, \mathscr{F}(F), \boldsymbol{\alpha})$ of the form $\mathscr{F}^j(F): F_o^j(F) \supset F_1^j(F) \supset 0, \alpha^j = (0, \alpha), 0 < \alpha < 1$. We also assume that for all j support of D_j is contained in the set of nonsingular points of X. Henceforth we restrict ourselves to bundles of the above type. We also assume that the base field is that of complex numbers.

DEFINITION 1.9

A morphism of GPS is a morphism of torsion-free sheaves $f: F \to F'$ such that $(f|D_j)(F_1^j(F)) \subseteq F_1^j(F')$ for all j.

Lemma 1.10. *Let $(F, \mathscr{F}(F), \boldsymbol{\alpha})$ be a semistable GPS. Then there exists an integer n_1 dependent only on g ($=$ arithmetic genus of X) and degree $D_j, j \in J$ such that if $\chi(F) = n > n_1$, then*

(1) $H^1(F) = 0, \mathbf{C}^n \approx H^0(F)$,
(2) *F is generated by global sections,*
(3) *$H^0(F) \to H^0(F \otimes \mathcal{O}_{D_j})$ is onto.*

Proof. This follows from $H^1(F') \approx H^0(X, \mathrm{Hom}(F', \omega_X))^*$ and the latter is zero if $p\mu(F')$ is sufficiently large (depending on $\chi(F'), g$). For (3) we need to take $F' = F(-D_j)$. (For details, see Lemma 3.7 [1]).

Lemma 1.11. *A morphism f of semistable GPS of same par μ is of constant rank. If the GPS have the same rank and one of them is stable, than either $f = 0$ or f is an isomorphism.*

Proof. This can be proved similarly as in Lemma 3.8 [1].

COROLLARY 1.12.

A stable GPS is simple i.e. its only endomorphisms are homothesies.

PROPOSITION 1.13.

The category S of all semistable GPS on X (of type described in 1.18) *with a fixed par $\mu = m$ is an abelian category. Its simple objects are the stable GPS.*

Proof. This follows from 1.11 and 1.12.

DEFINITION 1.14

In view of the above proposition, a semistable GPS$(E,\mathscr{F},\alpha)$ in S has a filtration with successive quotients stable GPS with par $\mu = m$. We denote by gr$(E,\mathscr{F},\alpha)$ the associated graded object for the filtration. Up to isomorphism this object is independent of the choice of stable filtration. Define an equivalence relation on S by $(E,\mathscr{F},\alpha)$ is equivalent to $(E',\mathscr{F}',\alpha)$ iff gr$(E,\mathscr{F},\alpha) \approx$ gr$(E',\mathscr{F}',\alpha)$.

Remark 1.5. We may (for convenience) use the terminology 'a GPS E'' when there is no confusion about parabolic structure possible.

2. Construction of the moduli space

2.1 Consider semistable GPS F of type described in 1.8 with rank k, Euler characteristic $n > n_1$ fixed. Let $P(m)$ be the Hilbert polynomial of F. Let $Q = Q(\mathcal{O}^n, P(m))$ be the quot scheme of coherent sheaves over X which are quotients of $\mathcal{O}^n$ and have Hilbert polynomial equal to P. Let $\mathscr{F}$ denote the universal quotient sheaf on $Q \times X$. Let R be the open subscheme of Q consisting of points $q \in Q$ such that $\mathscr{F}_q = \mathscr{F}|q \times X$ is torsion-free and the map $H^0(\mathcal{O}^n) \to H^0(\mathscr{F}_q)$ is an isomorphism. It follows that $H^1(\mathscr{F}_q) = 0$ for $q \in R$. For every j, let $p_j{:}R \times D_j \to R$ be the canonical map and define $V_j = (p_{j^*})(\mathscr{F}_{|D_j})$. Let $G(V_j)$ be the flag bundle over R of the type determined by the parabolic structure over D_j. It is a relative Grassmannian bundle of quotients of rank q_j. Let $\tilde{R}$ denote the fibre product of $\{G(V_j)\}_j$ over R. Let $\tilde{R}^s$ (resp. $\tilde{R}^{ss}$) denote the subset of $\tilde{R}$ corresponding to stable (resp. semistable) GPS. Similarly we can define $\tilde{Q}$, $\tilde{Q}^s$ and $\tilde{Q}^{ss}$.

The quot scheme Q has a natural embedding in a Grassmannian. For $m \geqslant M_1(n)$ the natural map $H^0(\mathcal{O}_X^n(m)) \to H^0(\mathscr{F}_q(m))$ is surjective for all $q \in Q$. Let $W = H^0(\mathcal{O}_X(m))$ then $H^0(\mathscr{F}_q(m))$ is a quotient of $\mathbf{C}^n \otimes W$ of dimension $P(m)$ for $m \geqslant M_1$. This gives a closed embedding $Q \to \mathrm{Grass}_{p(m)}(\mathbf{C}^n \otimes W)$. A point q of $\tilde{Q}$ gives for each j, a q_j-dimensional quotient of $\mathbf{C}^n$. Hence we get an embedding $\tilde{Q} \to \mathbf{Z} = \mathrm{Grass}_{P(m)}(\mathbf{C}^n \otimes W) \times (\times_j \mathrm{Grass}_{qj}(\mathbf{C}^n))$. This embedding is equivariant under the action of PGL(n). The PGL(n) action on $\tilde{Q}$ and $\mathbf{C}^n$ is the natural one, while on W it acts trivially. On $\mathbf{Z}$ we take the polarization

$$a(n - \alpha \sum_j q_j)/km \times a\alpha \times \cdots \times a\alpha,$$

where 1 denotes $\mathcal{O}(1)$ and a is a sufficiently big integer to make all the numbers above integers, $n > \Sigma_j q_j$.

We denote a point of Z by $(P,(P_j)_j)$ where $P{:}\mathbf{C}^n \otimes W \to U, P_j{:}C^n \to U_j$ are surjective maps, $\dim U = P(m), \dim U_j = q_j$ for all j. Similarly a point of $\tilde{Q}$ is denoted by $(p,(p_j)_j)$

where $p:\mathcal{O}^n \to F, p_j: H^0(F|D_j) \to Q_j^F$ are surjections, $\dim Q_j^F = q_j \forall j$. For a subsheaf E of F, we define $Q_j^E = p_j(H^0(E|D_j))$. For a quotient $q: F \to G$, define $Q_j^G = H^0(G|D_j)/q(\operatorname{Ker} p_j)$. For simplicity of notation, we denote $\dim(Q_j^E)$ ($\dim(Q_j^G)$) by $q_j(E)$ (by $q_j(G)$). In particular, $q_j = q_j(F)$.

PROPOSITION 2.2

For a nontrivial proper subspace $H \subset \mathbf{C}^n$ of dimension h define σ_H by

$$\sigma_H = \left(\left(n - \alpha \sum_j q_j\right) \Big/ km\right)(hP(m) - n \dim P(H \otimes W)) + \alpha \sum_j (q_j h - n \dim P_j(H)).$$

Then a point $(P, (P_j))$ of Z is semistable (resp. stable) for $PGL(n)$ — action (with the above polarization) if and only if $\sigma_H \leqslant 0$ (respectively < 0).

Proof. See [3, Proposition 5.1.1] and [2, Proposition 4.3].

DEFINITION 2.3

Let F be a torsion-free sheaf of rank k on X. For every subsheaf E of F and $m \geqslant 0$ integer define

$$\chi_E(m) = \left(\left(n - \alpha \sum_j q_j\right) \Big/ km\right)(\chi(E)P(m) - n\chi(E(m)) + \alpha \sum_j (q_j \chi(E) - n q_j(E)),$$

$$\sigma_E(m) = \left(\left(n - \alpha \sum_j q_j\right) \Big/ km\right)(h^0(E)P(m) - n\chi(E(m)) + \alpha \sum_j (q_j h^0(E) - n q_j(E)).$$

Lemma 2.4. *Let F be a torsion-free sheaf corresponding to a point $(p, (p_j)_j)$ of $\tilde{Q}$. Then F is semistable (respectively stable) if and only if for every subsheaf E of F we have $\chi_E = \chi_E(m) \leqslant$ (resp. < 0) for any integer m.*

Proof. Let E be a subsheaf of F with F/E torsion-free. Substituting $P(m) = km + n$, $\chi(E(m)) = \chi(E) + mr(E)(r(E) = \text{rank of } E)$ in the expression for χ_E and simplifying one gets

$$\chi_E(m) = nr(E)\left[\chi(E)/r(E) - n/k + \alpha \sum_j q_j/k - \alpha \sum_j q_j(E)/r(E)\right].$$

By definition F is semistable (respectively stable) if and only if the expression in the square bracket is $\leqslant 0$ (resp. < 0).

Suppose now that F/E is not torsion-free. Then there exists $\tilde{E}, E \subset \tilde{E}$ such that $F/\tilde{E}$ is torsion-free, rank $E =$ rank $\tilde{E}$. Let $\tilde{E}/E = \tau = \tilde{\tau} + \Sigma_j \tau_j$, where $\tau_{jD_j} = \tau_{j|D_j}$. By the above argument, $\chi_{\tilde{E}}(m) \leqslant 0$. We claim that $\chi_E(m) < \chi_{\tilde{E}}(m)$. Using $\chi(\tilde{E}(m)) - \chi(E(m)) = h^0(\tau)$ for $m \geqslant 0$, $q_j(\tilde{E}) - q_j(E) \leqslant h^0(\tau_j)$ we get $\chi_E(m) - \chi_{\tilde{E}}(m) = n(\alpha \Sigma_j h^0(\tau_j) - h^0(\tau)) < 0$ since $\alpha < 1$.

Lemma 2.5. *There exists an integer $M_2(n) \geqslant M_1(n)$ such that if $(p,(p_j)) \in \tilde{Q}$ is a point satisfying the following two conditions, then the image of the point in Z is semistable (resp. stable).*

(1) *The canonical map $\mathbf{C}^n \to H^0(F)$ is an isomorphism.*

(2) *For every subsheaf E of F generated by global sections, $\sigma_E(m) \leqslant 0$ (resp. < 0) for $m \geqslant M_2(n)$.*

Proof. Let $H \subset \mathbf{C}^n$ be a subspace. Let E be the subsheaf of F generated by H and let K be the kernel of the surjection $H \otimes \mathcal{O}_X \to E$. As H varies over subspaces of $\mathbf{C}^n$ and F varies over Q, the sheaves E and hence K form a bounded family. Hence there exists $M_2(n)$ such that for $m \geqslant M_2(n), h^1(E(-D_j)(m)) = 0$, $h^1(E(m)) = 0$ and $h^1(K(m)) = 0$ for all such E and K. It follows that dim $P(H \otimes W) = \chi(E(m))$. Clearly dim $H \leqslant h^0(E)$, $\dim P_j(H) = q_j(E)$. Therefore $\sigma_H \leqslant \sigma_E(m) \leqslant 0$ (resp. < 0). Thus the image $(P,(P_j))$ of $(p,(p_j))$ is semistable (resp. stable).

Lemma 2.6. *One can find $M_3(n) \geqslant M_2(n)$ such that for $m \geqslant M_3(n)$ the following holds. If $(p, p_j)) \in \tilde{Q}$ is a point whose image in Z is semistable then* (i) *$\mathbf{C}^n \to H^0(F)$ is injective and* (ii) *for all torsion-free quotients $F \to G \to 0$, one has $\tau_G \leqslant 0$. Here τ_G is defined by*

$$\tau_G = \left(\frac{n - \alpha\Sigma_j q_j}{k}\right)\left(-kh^0(G) + nr(G)\right) + \alpha\sum_j (nq_j(G) - q_j h^0(G)).$$

Proof. Note that if H_0 is the kernel of the map $\mathbf{C}^n \to H^0(F)$, then $\sigma_{H_0} > 0$ contradicting the semistability of the image point in Z (Proposition 2.2). Hence (i) follows. For (ii), suppose that there exists a torsion-free quotient G with $\tau_G > 0$. Then $h^0(G) < n$, for $h^0(G) \geqslant n$ implies $\tau_G \leqslant 0$. Let H be the kernel of the composite $\mathbf{C}^n \to H^0(F) \to H^0(G)$. Let E denote the subsheaf of F generated by H. Clearly we have $r(E) + r(G) \leqslant k$, $h^0(G) \geqslant n - h, q_j(G) \leqslant q_j - q_j(E)$, dim $P_j(H) \leqslant q_j(E)$. Substituting these in the expression for τ_G one gets

$$\left(\left(n - \alpha\sum_j q_j\right)\Big/ k\right)(kh - nr(E)) + \alpha\sum_j (q_j h - nq_j(E)) > 0.$$

Since H and hence E runs over a bounded family we can find $M_3(n) \geqslant M_2(n)$ such that for $m \geqslant M_3(n)$, the term $kh - nr(E)$ can be replaced by $(hP(m) - n\chi(E(m))/m = (hP(m) - n \dim P(H \otimes W))/m$. Thus we get $\sigma_H > 0$ contradicting the semistability of the image point in Z.

Lemma 2.7. *There exists $n_2 \geqslant n_1$ such that for all semistable GPS F with Euler characteristic $n \geqslant n_2$ the following holds*

(1) *If* $E \subset F$ *then* $\tau_E \leqslant 0$ *where*

$$\tau_E = \left(\left(n - \alpha \sum_j q_j\right) \Big/ k\right)(kh^0(E) - nr(E)) + \alpha \sum_j (q_j h^0(E) - nq_j(E)).$$

(2) *If* $\tau_E = 0$ *for some* $E \subset F$, *then* $\chi_E = 0$.
(3) *If* $\tau_E < 0$, *then* $\sigma_E(m) < 0$ *for* $m \geqslant M_4(n)$. *If* $\tau_E = 0$, *then* $\sigma_E(m) = 0$ *for* $m \geqslant M_4(n)$.

Proof. (1) Let $0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$ be the Harder-Narasimhan filtration of E considered as a torsion-free sheaf only, ignoring the parabolic structure. Let $Q_i = E_i/E_{i-1}$, $i = 1, \cdots, r, \mu_i =$ degree $Q_i/$ rank $Q_i, v = \inf \mu_i$. One has $\mu_i > \mu_{i+1} \forall i < r$ (by definition), $h^0(E) \leqslant \Sigma_j h^0(Q_i)$ (by induction). Using Corollary 2.5 [4], this implies $h^0(E) \leqslant \Sigma^i r(Q_i)$ $(\mu_i + B_1), B_i$ constant. Since Q_1 is a subsheaf of a semistable GPS F we have $\mu_i \leqslant \mu_1 \leqslant \mu(F) + w\forall_i, w = (wtF)/k$. Since $n \geqslant n_1, \Sigma_i r(Q_i) = r(E), v = \mu_{i_o}$ we get $h^0(E) \leqslant \Sigma_{i \neq i_o} r(Q_i)(\mu(F) + w + B_1) + (v + B_1) + (r(Q_{i_o}) - 1)(\mu(F) + w + B_1) \leqslant v + (r(E) - 1) \cdot n/k + B_2, B_2$ constant. Hence $\tau_E \leqslant n(v + B_2 - n/k)$. Therefore if $v \leqslant n/k - B_2$ (resp. $<$) then $\tau_E \leqslant 0$ (resp. < 0). We can choose n_2 large enough so that for $n \geqslant n_2$, we have $h^1(Q(m)) = 0$ for all $m \geqslant 0$ and for all stable torsion-free sheaves Q of rank $\leqslant k$ and $\mu \geqslant n/k - B_2$. Hence if $v \geqslant n/k - B_2$ for E, then $h^1(Q_i(m) = 0 \forall i$, therefore $h^1(E(m)) = 0$ and $\chi(E(m)) = h^0(E(m))$ for $m \geqslant 0$. Then $\tau_E = \chi_E$ and $\chi_E \leqslant 0$ by Lemma 2.4. Thus $\tau_E \leqslant 0$ for all $E \subset F$.

(2) If $\tau_E = 0$, then by the above argument one must have $v \geqslant n/k - B_2, \chi(E) = h^0(E)$ and $\tau_E = \chi_E$. Thus $\chi_E = \sigma_E(m) = 0$.

(3) Note that $\tau_E = \lim_{m \to \infty} \sigma_E(m)$. Hence given $\varepsilon > 0, \exists M_4(n)$ such that for $m \geqslant M_4(n)$, $\sigma_E(m) < \tau_E + \varepsilon$. If $\tau_E < 0$ then choosing ε such that $\tau_E + \varepsilon < 0$, we get $\sigma_E(m) < 0$ for $m \geqslant M_4(n)$.

Theorem 1. (I) *Let X be an integral projective curve of arithmetic genus g over* **C**. *Let* $\{D_j\}_{j \in J}$ *be finitely many effective Cartier divisors in X such that the support of D_j does not intersect the set of singular points of X for all j, supports of D_j are mutually disjoint and degree $D_j = d_j, j \in J$. Let S denote the set of equivalence classes of semistable GPS F of rank k degree d with parabolic structure over D_j given by* $F_o^j(F) = H^0(F \otimes \mathcal{O}_{Dj}) \supset F_1^j(F) \supset 0$, *co-dimension of $F_1^j(F)$ in $F_o^j(F)$ equal to q_j (fixed) for $j \in J$ and weights $(0, \alpha), 0 < \alpha < 1$. Then S has the structure of a projective variety $M(k,d)$ of dimension* $k^2(g-1) + 1 + \Sigma_j q_j(kd_j - q_j)$.

(II) *If X is nonsingular, then $M(k,d)$ is normal. If further $(k,d) = 1, q_j$ is a multiple of k and α is sufficiently near 1 then $M(k,d)$ is nonsingular and it is a fine moduli space.*

Proof. Let w_X denote the dualising sheaf of X, it is a torsion-free sheaf. Fix $n >$ max $(n_2, kh^0(w_X) + \alpha \Sigma_j q_j)$ and $m \geqslant M_4(n)$. We keep the notations of 2.1. We shall show that a geometric invariant theoretic quotient of $\tilde{R}$ modulo PGL(n) exists. Our required moduli space $M(k,d)$ will be this quotient. $\tilde{R}$ is an open subset of $\tilde{Q}, \tilde{Q}$ is embedded in Z (with m, n as above) by a PGL(n) equivariant embedding. We first claim that if $(p, p_j)) \in R^{ss}$ (resp. $\mathbf{R}^s$) then its image belongs to Z^{ss} (resp. Z^s). This follows immediately from Lemma 2.7 and Lemma 2.5. Let F correspond to a point in $\tilde{R}^{ss} - \tilde{R}^s$. Then F has a subsheaf E which is a torsion-free stable GPS with par $\mu(E) =$ par $\mu(F)$ i.e. $\chi_E = 0$. For such an $E, \sigma_{H^0(E)} = \chi_E$ (Lemma 1.10), hence the image in Z belongs to $Z^{ss} - Z^s$.

Conversely we shall now check that if a point in $\tilde{Q}$ is such that its image belongs to Z^{ss}, then the point is in $\tilde{R}^{ss}$ i.e. if F is the corresponding quotient, then F is torsion-free, the map $\mathbf{C}^n \to H^0(F)$ is an isomorphism and F is a semistable GPS. Lemma 2.6 implies that $\mathbf{C}^n \to H^0(F)$ is injective and for every rank 1 torsion-free quotient G of $F, n \leqslant kh^0(G) + \alpha \Sigma_j q_j$ (as $\tau_G \leqslant 0$). We claim that $H^1(F) = 0$. Otherwise there exists a nontrivial homomorphism $F \to w_X$. If G is the sheaf image of this morphism, $h^0(w_X) \geqslant h^0(G)$ and hence $n \leqslant kh^0(w_X) + \alpha \Sigma_j q_j$ contradicting the assumptions on n. Thus $h^0(F) = n$ and $\mathbf{C}^n \to H^0(F)$ is an isomorphism. Let τ be the torsion subsheaf of $F, \tau = \tau_o + \Sigma_j \tau_j$, support $\tau_j \subseteq \operatorname{supp} D_j$, $(\operatorname{supp} \tau_o) \cap (\cup \operatorname{supp} D_j) = \phi$. Taking $H = H^0(\tau_0), H^0(\tau_j), \sigma_H \leqslant 0$ gives $H^0(\tau_o) = 0, H^0(\tau_j) = 0$. Here $\alpha < 1$ is crucial since $\sigma_H = n(h - \alpha \dim P_j(H))$. Thus $H^0(\tau) = H^0(\tau_o) + \Sigma_j H^0(\tau_j) = 0$ i.e. $\tau = 0$.

Suppose that F is not semistable. Then there exists a subsheaf E of F such that E is a semistable GPS with $\operatorname{par} \mu(E) > \operatorname{par} \mu(F)$ i.e. $\chi_E > 0$. By Lemma 1.10, $\sigma_{H^0(E)} = \chi_E > 0$ contradicting the semistability of the image point in Z.

It follows that the (geometric invariant theoretic) quotient $M(k,d)$ of $\tilde{R}$ mod $\mathrm{PGL}(n)$ is the same as that of $\tilde{Q}$ and it exists if and only if the quotient of image of $\tilde{Q}$ in Z exists. It is well known that the latter exists. The quotient $M(k,d)$ is a projective variety as $\tilde{Q}$ is so. It is easy to check that the points of $M(k,d)$ correspond to equivalence classes of semistable GPS(3.15, [1]; [4]).

(2) If X is nonsingular $\tilde{R}$ is known to be nonsingular and hence $M(k,d)$ is normal. If $(k,d) = 1, \alpha$ is sufficiently near to 1 and q_j is an integral multiple of k, then GPS is semistable if and only if it is stable by Lemma 3.3 (or Lemma 3.17, [1]). The nonsingularity of $\tilde{R}$ together with corollary 1.12 then imply that $M(k,d)$ is nonsingular. One can show that $M(k,d)$ is then a fine moduli space, by proving the universal bundle on R descends to a universal bundle on $M(k,d)$ after twisting by a line bundle (see [1], Proposition 3.18).

3. Application

3.1. Let Y be an integral projective curve with only singularities ordinary double points $\{y_j\} j \in J$. Let $J' \subseteq J$ be a subset. Let X be the curve (proper transform) obtained by blowing up $\{y_j\}_{j \in J'}$. Let $\pi_{XY}: X \to Y$ be the natural morphism. Let D_j denote the divisor $\pi_{XY}^{-1}(y_j), j \in J'$. All the QPS and GPS that we consider are assumed to be of the type described in 1.8. We also assume that $\dim F_1^j(F)) = r(F)$ for $j \in J'$.

DEFINITION 3.2

Let α be a real number in $[0,1]$. A QPS $(F, \mathscr{F})$ on X is α-stable (resp. α-semistable) if for any proper subsheaf K of F with torsion-free quotient, one has

$$(d(K) + \alpha \sum_{j \in J'} \dim F_1^j(K))/r(K) < (\leqslant)(d(F) + \alpha J' r(F))/r(F).$$

Remark. For $0 < \alpha < 1$, the above condition is same as that for stability (resp. semistability) of the $\mathrm{GPS}(F, \mathscr{F}, \boldsymbol{\alpha})$ with $\alpha^j = (0, \alpha), j \in J'$.

Lemma 3.3. (1) *Suppose that* $1 - 1/J'r(F)(r(F)-1) < \alpha < 1$. *Then* $(F, \mathscr{F})$ *is* α-*semistable implies that it is* 1-*semistable. If the QPS is* 1-*stable then it is also* α-*stable.*

(2) *Assume that* $(r(F), d(F)) = 1$. *Then the QPS is* 1-*stable if and only if it is* 1-*semistable. Thus under the assumptions of* (1) and (2) 1-*stability*, α-*stability and* α-*semistability are all equivalent.*

Proof. This is a straightforward generalization of Lemma 3.17 [1].

PROPOSITION 3.4.

Let Q *denote the set of isomorphism classes of QPS* $(F, \mathscr{F})$ *on* X *of given type* (3.1). *Let* $r(F) = k, d(F) = d$ *be fixed. Let* S *be the set of isomorphism classes of torsion-free sheaves of rank* k *and degree* d *on* Y. *Let* S_k *denote the subset of* S *corresponding to sheaves which are locally free at* y_j *for* $j \in J'$. *Then (a) there is a surjective map* $f_{XY}: Q \to S$ *such that its restriction to* $f_{XY}^{-1}(S_k)$ *is a bijection onto* S_k. (b) $(F, \mathscr{F})$ *is* 1-*stable* (1-*semistable*) *iff its image under* f_{XY} *is stable (semistable).*

Proof. Let $D_j = x_j + z_j$. Then $((\pi_{XY})_* F) \otimes k(y_j) = (k(x_j) \oplus k(z_j))^{r(F)} = H^0(F \otimes \mathscr{O}_{Dj}) \equiv F_o^j(F)$. Thus we have a surjective $\mathscr{O}_Y$-linear map $(\pi_{XY})_* F \to F_o^j(F)$. Let F' be the kernel of the composite of this map with the surjection $F_o^j(F) \to F_o^j(F)/F_1^j(F)$. Since $d(F) = \chi(F) - r(F)\chi(\mathscr{O}_X), d(F') = \chi(F') - r(F')\chi(\mathscr{O}_Y), \chi(F) = \chi((\pi_{XY})_* F), \chi(\mathscr{O}_Y) = \chi(\mathscr{O}_X) - J'$, it follows that if $(F, \mathscr{F}) \in Q$ then $F' \in S$. We define $f_{XY}(F, \mathscr{F})) = F'$. If $F' \in S_k$ then $F = \pi_{XY}^* F'$ and $F_1^j(F) = F' \otimes k(y_j) \subset F_o^j(F)$ gives the bijection. Surjectivity of f can be proved as in 4.5 [1] while the last assertion follows exactly as in 4.2 [1].

Theorem 2. (I) *Let* $M_{X,J'}$ *be the moduli space of semistable GPS on* X *of type described in* 3.1. *Assume that* α *satisfies the conditions of Lemma* 3.3(1). *Then there is a surjective birational morphism* $f_{XY}: M_{X,J'} \to M_{Y,\phi}$ (= *moduli space of torsion-free sheaves on* Y).

(II) *Let* Z *be the desingularization of* Y. *Then the morphism* $f_{ZY}: M_{Z,J} \to M_{Y,\phi}$ *factors as* $f_{XY} \circ f_{ZX}$. *If the conditions of Lemma 3.3 are satisfied then* $M_{Z,J}$ *is a desingularization of* $M_{Y,\phi}$ *and* $M_{X,J'} (J' \subset J)$ *are 'partial desingularizations.*

Proof. (I) This follows easily from Lemma 3.3 and Proposition 3.4 since it is easy to globalise the construction (of f_{XY}) to families of GPS. (See Theorem 2 [1] for details).

(II) Let $f_{ZX}(F, \mathscr{F}) = F'$. Notice that π_{ZX} is an isomorphism outside $J - J'$. Hence F' has a parabolic structure $\mathscr{F}'$ over $D_j, j \in J'$ viz. $F_i^j(F') \approx F_i^j(F)$ for $i = 0, 1, j \in J'$. Thus $f_{ZX}(F, \mathscr{F}) = (F', \mathscr{F}') \in M_{X,J'}$. Let $f_{XY}(F', \mathscr{F}') = F''$. Then we have the exact sequences (defining $F', \mathscr{F}''$)

$$0 \to F' \to (\pi_{ZX})_* F \to \oplus_{J-J'} F_o^j(F)/F_1^j(F) \to 0$$

$$0 \to F'' \to (\pi_{XY})_* F' \to \oplus_{J'} F_o^j(F')/F_1^j(F') \to 0.$$

Using these and $\pi_{ZY} = \pi_{XY}\pi_{ZX}$, one gets

$$0 \to F'' \to (\pi_{ZY})_* F \to \oplus_J F_o^j(F)/F_1^j(F) \to 0$$

proving $f_{ZY} = f_{XY} \circ f_{ZX}$. The last assertion follows from Theorem 1 (II).

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Non-existence of nodal solution for m-Laplace equation involving critical Sobolev exponents

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Abstract. In this paper we study the non-existence of nodal solutions for critical Sobolev exponent problem

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{m-2}\nabla u) &= |u|^{p-1}u + |u|^{q-1}u \text{ in } B(R)\\ u &= 0 \qquad \text{on } \partial B(R) \end{aligned}$$

where $B(R)$ is a ball of radius R in $\mathbb{R}^n$.

Keywords. Critical exponent; eigenvalue; m-Laplacian.

1. Introduction

Consider the problem

$$\left.\begin{aligned} -\Delta_m u &= |u|^{p-1}u + |u|^{q-1}u \text{ in } B(R)\\ u &= 0 \qquad \text{on } \partial B(R), \end{aligned}\right\} \tag{1.1}$$

where $B(R)$ is a ball in $\mathbb{R}^n$ of radius R, $\Delta_m u = \operatorname{div}(|\nabla u|^{m-2}\nabla u)$ and $1 < m < n$, $p+1 = mn/(n-m)$ is the critical Sobolev exponent for the non-compact imbedding $H_0^{1,m} \to L^{p+1}$ and $1 \leqslant q \leqslant p-1$. In this paper we are interested in the radial solutions of (1.1) which change sign.

For $m=2$, this problem has been discussed by many authors. It has been shown by Cerami *et al* [7] and Solomini [13] that (1.1) admits infinitely many radial solutions which change sign for $q=1$ and $n \geqslant 7$. Atkinson *et al* [4, 5] and Adimurthi and Yadava [1] have proved that this result of infinitely many nodal solutions is optimal in the sense, when $3 \leqslant n \leqslant 6$, $q=1$, then (1.1) does not admit any radial solution which changes sign for all R sufficiently small. For $p-1 < q < p$, Jones [11] and Atkinsion-Peletier [3] have proved that (1.1) admits infinitely many radial solutions which change sign. It has been shown by Jones [11] for $1 < q < p-1$ and by Knaap [12] for $q = p-1$ that (1.1) does not admit any radial solution which changes sign provided R is sufficiently small. Atkinson *et al* [4, 5] have used asymptotic analysis to prove their non-existence result and Jones [11] has adapted dynamical system approach. In [1], the non-existence result has been obtained by Pohozaev's identity.

In this paper, following the method used in [1], we extend the non-existence result for the general m, $1 < m < n$. We prove

Theorem. *Let $m-1 \leqslant q \leqslant p-1$. Then there exists a $R_0 > 0$ such that for all $0 < R < R_0$,*

$$\left.\begin{array}{ll} -\Delta_m u = |u|^{p-1}u + |u|^{q-1}u & \text{in } B(R) \\ u = 0 & \text{on } \partial B(R) \end{array}\right\} \tag{1.2}$$

does not admit any radial solution which changes sign.

Remark 1. If $m = 2$, the above theorem gives all the above mentioned known results of the non-existence of solution which changes sign.

Remark 2. For $0 < q < m-1$, it has been shown in [9] that for sufficiently small R, (1.2) admits infinitely many solutions.

Remark 3. If $q = m-1$, then the above theorem is true for the range $m < n \leqslant m^2 + m$. For $m < n < m^2$, Atkinson *et al* [6] have proved a more stronger result, namely (1.2) does not admit any positive radial solution for R sufficiently small.

2. Proof of the theorem

Since we are looking for radial solution, we can set $u = u(r)$, $r = |x|$ and write (1.2) as

$$\left.\begin{array}{l} -\dfrac{\mathrm{d}}{\mathrm{d}r}(r^{n-1}|u'|^{m-2}u') = r^{n-1}(|u|^{p-1} + |u|^{q-1})u \quad \text{in } (0, R) \\ u'(0) = u(R) = 0. \end{array}\right\} \tag{2.1}$$

To study the problem (2.1), we can consider the associated initial value problem

$$\left.\begin{array}{l} -\dfrac{\mathrm{d}}{\mathrm{d}r}(r^{n-1}|v'|^{m-2}v') = r^{n-1}(|v|^{p-1} + |v|^{q-1})v \quad \text{in } (0, \infty) \\ v'(0) = 0, \quad v(0) = \gamma. \end{array}\right\} \tag{2.2}$$

Let $v(r, \gamma)$ be the unique solution of (2.2). Let $0 < R_1(\gamma) < R_2(\gamma) < \ldots$ be the zeros of $v(r, \gamma)$. In order to prove the Theorem, it is enough to show that there exists a $C_0 > 0$ such that

$$R_2(\gamma) \geqslant C_0 \tag{2.3}$$

for all $\gamma \in (0, \infty)$. To prove (2.3) we need the following.

Lemma. We have

$$\sup_{\gamma \in (0,\infty)} \{|v(r,\gamma)|; R_1(\gamma) \leqslant r \leqslant R_2(\gamma)\} \leqslant k_0 \tag{2.4}$$

where

$$k_0 = \left.\begin{array}{ll} \dfrac{1}{p} & \text{if } q = p-1 \\ \dfrac{(p-q-1)^{(p-q-1)/(p-q)}}{(q+1)} & \text{if } q < p-1. \end{array}\right\} \tag{2.5}$$

Proof. Suppose (2.4) is not true. Then there exists as $\gamma > 0$ and a $k > k_0$ such that $|v(r,\gamma)| = k$ has a solution in $[R_1(\gamma), R_2(\gamma)]$. Let $R > R_1(\gamma)$ be the first point at which $v(R,\gamma) = -k$. Let $w(r) = v(r,\gamma) + k$. Then w satisfies

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}r}(r^{n-1}|w'|^{m-2}w') &= r^{n-1}f(w) \quad \text{in } (0,R), \\ w &> 0 \\ w'(0) = w(R) &= 0. \end{aligned}\right\} \tag{2.6}$$

where $f(w) = (|w-k|^{p-1} + |w-k|^{q-1})(w-k)$. Let F denote the primitive of f. Then by Pohozaev's identity [8] and [10], we have

$$\begin{aligned} 0 &\leqslant \int_0^R \left\{\left(\frac{mn}{n-m}\right)F(w) - f(w)w\right\} r^{n-1}\,\mathrm{d}r \\ &= \int_0^R g(w-k)r^{n-1}\,\mathrm{d}r - \left\{k^{p+1} + \left(\frac{p+1}{q+1}\right)k^{q+1}\right\}\frac{R^n}{n}, \end{aligned} \tag{2.7}$$

where

$$g(s) = \left(\frac{p-q}{q+1}\right)|s|^{q+1} - k|s|^{p-1}s - k|s|^{q-1}s.$$

Now observe that for $-k \leqslant s \leqslant 0$, $g(s)$ is decreasing and non-negative. Therefore

$$g(s) \leqslant g(-k) = k^{p+1} + \left(\frac{p+1}{q+1}\right)k^{q+1} \tag{2.8}$$

for all $s \in [-k, 0]$.

For $s > 0$ we have

Claim. $g(s) < 0$ for all $s > 0$.

For $s > 0$, let $h(s) = g(s)/s^q$. Then

$$h(s) = \left(\frac{p-q}{q+1}\right)s - ks^{p-q} - k.$$

Case 1. Let $q = p - 1$. Then

$$h(s) = -\left(k - \frac{1}{p}\right)s - k.$$

Since $k > 1/p$, we get $h(s) < 0$.

Case 2. Let $q < p - 1$. Then h has a maximum at

$$s_0 = \left(\frac{1}{k(q+1)}\right)^{1/(p-q-1)}$$

Since $k > k_0$, we get

$$h(s_0) = \frac{p-q-1}{(q+1)^{p-q/(p-q-1)}} \frac{1}{k^{1/(p-q-1)}} - k < 0$$

and this proves the claim.

Now from (2.7), (2.8) and Claim, we have

$$\begin{aligned} 0 &\leqslant \int_{0 \leqslant w \leqslant k} g(w-k) r^{n-1} \, \mathrm{d}r + \int_{w>k} g(w-k) r^{n-1} \, \mathrm{d}r \\ &\quad - \left\{ k^{p+1} + \left(\frac{p+1}{q+1}\right) k^{q+1} \right\} \frac{R^n}{n} \\ &< g(-k) \frac{R^n}{n} - \left\{ k^{p+1} + \left(\frac{p+1}{q+1}\right) k^{q+1} \right) \frac{R^n}{n} \\ &= 0 \end{aligned}$$

which is a contradiction. This proves the lemma.

Before going into the proof of (2.3), we recollect some known results about the first eigenvalue for Δ_m (see [2]).

Let Ω be a bounded domain with $C^{2,\beta}$ boundary and let $\alpha \in L^\infty(\Omega)$ be such that meas $\{x \in \Omega; \alpha(x) > 0\} \neq 0$. Then there exists a unique $\lambda(\alpha, \Omega) > 0$ such that

$$\begin{aligned} -\Delta_m \phi &= \lambda(\alpha, \Omega) \alpha |\phi|^{m-2} \phi \quad \text{in } \Omega \\ \phi &> 0 \\ \phi &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.9}$$

admits a unique (up to multiplication by a constant) solution.

Obviously, if $0 \leqslant \alpha_1 \leqslant \alpha_2$ and $\alpha_i \in L^\infty(\Omega)$, then

$$\lambda_1(\alpha_1, \Omega) \geqslant \lambda_1(\alpha_2, \Omega). \tag{2.10}$$

Moreover,

$$\lambda_1(1, \Omega) \to \infty \text{ as meas } (\Omega) \to 0. \tag{2.11}$$

Proof of (2.3). We claim that there exists a $\delta > 0$ such that

$$R_2(\gamma) - R_1(\gamma) \geqslant \delta \tag{2.12}$$

for all $\gamma \in (0, \infty)$.

Since $m - 1 \leqslant q$, by the lemma there exists a $C > 1$ such that

$$\sup_{\gamma \in (0,\infty)} \{ |v|^{p-m+1} + |v|^{q-m+1}; R_1(\gamma) \leqslant r \leqslant R_2(\gamma) \} < C. \tag{2.13}$$

Now suppose (2.12) is not true. Choose a $\gamma_0 > 0$ such that

$$\lambda_1(C, B(R_1(\gamma_0), R_2(\gamma_0))) \geqslant 2, \tag{2.14}$$

where $B(R_1(\gamma_0), R_2(\gamma_0)) = \{x \in \mathbb{R}^n; R_1(\gamma_0) \leqslant |x| \leqslant R_2(\gamma_0)\}$. On the other hand, from (2.2),

$$\lambda_1(|v|^{p-m+1} + |v|^{q-m+1}, B(R_1(\gamma_0), R_2(\gamma_0)) = 1. \tag{2.15}$$

This, together with (2.13) and (2.10), contradicts (2.14). This completes the proof of the Theorem.

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Solution of convex conservation laws in a strip

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Abstract. In this paper we consider scalar convex conservation laws in one space variable in a strip $D = \{(x,t)\colon 0 \leqslant x \leqslant 1,\ t > 0\}$ and obtain an explicit formula for the solution of the mixed initial boundary value problem, the boundary data being prescribed in the sense of Bardos-Leroux and Nedelec. We also get an explicit formula for the solution of weighted Burgers equation in a strip.

Keywords. Conservation laws; boundary value problem; explicit formula.

1. Introduction

We consider mixed initial boundary value problem for scalar convex conservation laws of the form

$$u_t + f(u)_x = 0 \tag{1.1}$$

in a strip $D = \{(x,t)\colon 0 \leqslant x \leqslant 1,\ t \geqslant 0\}$ with initial condition

$$u(x,0) = u_0(x). \tag{1.2}$$

The boundary conditions are prescribed in the sense of Bardos *et al* [1]. Let $u_1(t)$ and $u_2(t)$ are any bounded measurable functions, then this condition requires $u(0+,t)$ and $u(1-,t)$ to satisfy the following:

$$\sup_{k \in I(u(0+,t),\, u_1(t))} [\operatorname{sgn}(u(0+,t)-k)(f(u(0+,t)) - f(k))] = 0 \tag*{$(1.3)_0$}$$

$$\inf_{k \in I(u(1-,t),\, u_2(t))} [\operatorname{sgn}(u(1-,t)-k)(f(u(1-,t)) - f(k))] = 0 \tag*{$(1.3)_1$}$$

where for any real numbers a and b, $I(a,b)$ denotes the closed interval $[\min(a,b), \max(a,b)]$.

We assume the flux function $f(u)$ satisfies the following two conditions

$$f''(u) > 0 \tag*{(A_1)}$$

and

$$\lim_{|u|\to\infty} \frac{f(u)}{|u|} = \infty. \tag*{(A_2)}$$

Under the assumption (A_1), it can be easily checked that $(1.3)_0$ and $(1.3)_1$ are equivalent to $(1.3)'_0$ and $(1.3)'_1$ respectively, see Lefloch [3].

$$\left.\begin{array}{l} u(0+,t)=u_1^+(t) \\ \qquad\text{or} \\ f'(u(0+,t))\leqslant 0 \text{ and } f(u(0+,t))\geqslant f(u_1^+(t)) \end{array}\right\}\text{a.e } t>0, \tag{1.3$'_0$}$$

$$\left.\begin{array}{l} u(1-,t)=u_2^-(t) \\ \qquad\text{or} \\ f'(u(1-,t))\geqslant 0 \text{ and } f(u(0-,t))\geqslant f(u_2^-(t)) \end{array}\right\}\text{a.e } t>0, \tag{1.3$'_1$}$$

where

$$\left.\begin{array}{l} u_1^+(t)=\max\{u_1(t),\lambda\}, \\ u_2^-(t)=\min\{u_2(t),\lambda\}. \end{array}\right\}. \tag{1.4}$$

Here λ is the unique solution of the equation $f'(u)=0$. Because of the assumption (A_1) on $f(u)$, this λ satisfies

$$f(\lambda)=\min_{u\in R^1} f(u).$$

In order to have uniqueness of solution for (1.1), it is known that, an additional condition called entropy condition should be imposed. Under the condition (A_1) on $f(u)$ this condition requires $u(x+,t)$ and $u(x-,t)$ to satisfy

$$u(x-,t)\geqslant u(x+,t) \tag{1.5}$$

for every $0<x<1$, $t>0$.

Existence and uniqueness of solution of (1.1), (1.2), $(1.3)'_0$, $(1.3)'_1$, (1.4) and (1.5) follows from the work of Bardos *et al* [1], where they consider a more general problem in several space variable. In this paper we are interested in obtaining an explicit formula for the solution in the case of one space variable and $f(u)$ satisfying conditions (A_1) and (A_2). The formula we derive here is an extension to the mixed initial boundary case of an explicit formula derived by Lax [5], for the pure initial value problem. In the case of one boundary, i.e., when $D=\{x,t):x\geqslant 0, t\geqslant 0\}$, this problem was studied by Lefloch [3] and Joseph and Gowda [4]. Hamilton-Jacobi equation with Neumann type boundary condition was studied by Lions [6]. In one space variable they are closely related to conservation laws with Dirichlet boundary condition.

This paper is organized as follows. In § 2, we state the main result: In § 3, we give a detailed proof of the main theorem and in § 4, we study the weighted Burgers equation.

2. Statement of the main theorem

Before the statement of our main theorem we introduce some notations. For each fixed (x,y,t), $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$, $t>0$, $|i-j|\leqslant 1$, $i,j=0,1,2,\ldots$, $\mathscr{C}_{i,j}(x,y,t)$ denotes the following class of paths β in the strip

$$D=\{(z,s):0\leqslant z\leqslant 1,\ s\geqslant 0\}.$$

Each path connects the point $(y,0)$ to (x,t) and is of the form $z=\beta(s)$ where $\beta(s)$ is piecewise linear function which are straight lines in the interior of D, and having i straight line pieces lie on $x=0$ and j of them lie on $x=1$. For the cases $(i,j)=(0,0)$, $(i,j)=(2,1)$, $(i,j)=(1,2)$ see figures 1a, 1b and 1c respectively.

Denote

$$\mathscr{C}(x,y,t)=\bigcup_{\substack{i\geqslant 0,\, j\geqslant 0\\ |i-j|\leqslant 1}}\mathscr{C}_{ij}(x,y,t).$$

Let $f^*(u)$ is the convex dual of $f(u)$.

$$f^*(u)=\max_{\theta}\,[\theta u-f(\theta)]. \tag{2.1}$$

Let $u_0(x)\in L^\infty(0,1)$ and $u_1(t)$ and $u_2(t)$ are continuous bounded functions and let $u_1^+(t)$ and $u_2^-(t)$ be defined by (1.4). Let (x,y,t) be kept fixed. For each $\beta\in\mathscr{C}(x,y,t)$, we define

$$J(\beta)=-\int_{\{s:\beta(s)=0\}}f(u_1^+(s))\mathrm{d}s-\int_{\{s:\beta(s)=1\}}f(u_2^-(s))\mathrm{d}s+\int_{\{s:0<\beta(s)<1\}}f^*\left(\frac{\mathrm{d}\beta}{\mathrm{d}s}\right)\mathrm{d}s. \tag{2.2}$$

We let

$$Q(x,y,t)=\min_{\beta\in\mathscr{C}(x,y,t)}J(\beta). \tag{2.3}$$

We will see later that $Q(x,y,t)$ is Lipschitz continuous w.r.t. (x,y,t). Denote

$$Q_1(x,y,t)=\partial_x Q(x,y,t). \tag{2.4}$$

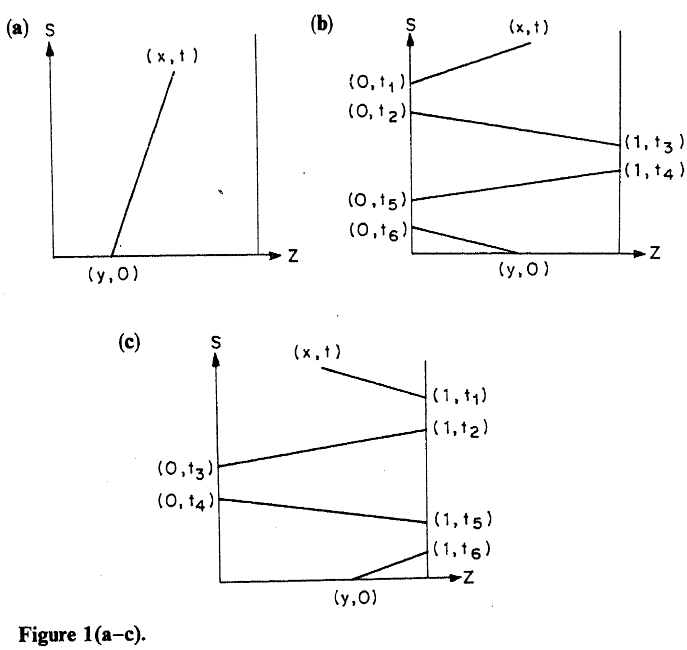

Figure 1(a–c).

We will see that for a.e. (x, t) there exists only one $y_0(x, t)$ which minimises

$$\min_{0\leqslant y\leqslant 1}\left[\int_0^y u_0(z)\mathrm{d}z + Q(x, y, t)\right] \tag{2.5}$$

We shall prove the following theorem.

Main theorem. *Let $u(x,t)$ be defined by*

$$u(x,t) = Q_1(x, y_0(x,t), t), \tag{2.6}$$

where $Q_1(x,y,t)$ is defined by (2.4) *and $y_0(x,t)$ minimizes* (2.5). *Then* (i) *$u(x,t)$ satisfies $u_t + f(u)_x = 0$, in the sense of distributions and satisfies the initial condition* (1.2) (ii) *$u(0+,t)$ and $u(1-,t)$ exists a.e. and satisfies the boundary conditions* $(1.3)'_0$ *and* $(1.3)'_1$. (iii) *For each fixed $t > 0$, $0 < x < 1$, $u(x \pm 0, t)$ exists and satisfies the entropy condition* (1.5).

3. Proof of the main theorem

The proof of the main theorem is broken up into several steps formulated as Lemmas. First we need some preliminaries. By definition any curve β in $\mathscr{C}_{i,j}(x,y,t)$ starts at $(y, 0)$ and ends at (x, t) and is made up of straight lines joined together at point of the boundary: $x = 0$ or $x = 1$. Let a curve β is given and let $(\beta(t_1), t_1)$, $(\beta(t_2), t_2)\ldots$ be the corners, *i.e.* the point of intersection of two straight lines of β. We assume that $t_1, t_2, t_3\ldots$ are ordered such that

$$t \geqslant t_1 > t_2 > .. > 0.$$

Note that $\beta(t_j)$ can take either 0 or 1 only, see figures 1a, 1b and 1c.

If (x, t) is a point on the boundary *i.e.* if $x = 0$ or $x = 1$, and let $\beta \in \mathscr{C}_{i,j}(x, y, t)$, by convention we take $t_1 = t$ iff for some $\varepsilon > 0$, $(t - \varepsilon, t) \subset \{s : \beta(s) = x\}$, see figure 2.

For $l = 0, 1$, we define

$$\mathscr{C}^l_{i,j}(x,y,t) = \{\beta \in \mathscr{C}_{i,j}(x,y,t) : \beta(t_1) = l\}.$$

Clearly, for $k = 0, 1, 2, \ldots$

$$\left.\begin{aligned}
&\mathscr{C}^0_{k+1,k}(x,y,t) = \mathscr{C}_{k+1,k}(x,y,t),\ \mathscr{C}^1_{k+1,k}(x,y,t) = \phi,\\
&\mathscr{C}^0_{k,k+1}(x,y,t) = \phi,\ \mathscr{C}^1_{k,k+1}(x,y,t) = \mathscr{C}_{k,k+1}(x,y,t),\\
&\mathscr{C}_{k,k}(x,y,t) = \mathscr{C}^0_{k,k}(x,y,t) \bigcup \mathscr{C}^1_{k,k}(x,y,t),\\
&\mathscr{C}(x,y,t) = \bigcup_{k=0}^{\infty} \{\mathscr{C}^0_{k,k}(x,y,t) \bigcup \mathscr{C}^1_{k,k}(x,y,t) \bigcup \mathscr{C}^1_{k,k+1}(x,y,t)\\
&\qquad \bigcup \mathscr{C}^0_{k+1,k}(x,y,t)\}.
\end{aligned}\right\} \tag{3.2}$$

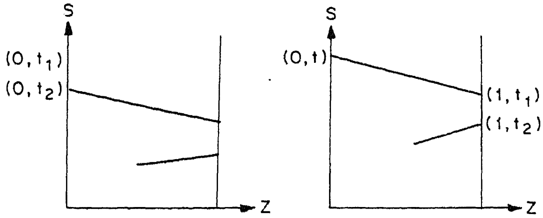

Figure 2.

For $\beta \in \mathscr{C}^0_{k+1,k}(x, y, t)$, we have

$$J(\beta) = J^0_{k+1,k}(x, y, t, t_1, \dots t_{4k+2}\},$$

where

$$\left.\begin{aligned} J^0_{k+1,k}(x, y, t, t_1, t_2, \dots t_{4k+2}) = -\sum_{j=0}^{k} \left\{ \int_{t_{4j+3}}^{t_{4j+1}} f(u_1^+(s))\mathrm{d}s \right\} \\ -\sum_{j=0}^{k-1} \left\{ \int_{t_{4j+4}}^{t_{4j+3}} f(u_2^-(s))\mathrm{d}s \right\} + (t - t_1)f^*\left(\frac{x}{t - t_1}\right) \\ + \sum_{j=1}^{2k} \left\{ (t_{2j} - t_{2j+1}) f^*\left(\frac{1}{t_{2j} - t_{2j+1}}\right) \right\} + t_{4k+2} f^*\left(\frac{-y}{t_{4k+2}}\right). \end{aligned}\right\} \quad (3.3)$$

For $\beta \in \mathscr{C}^1_{k,k+1}(x, y, t)$, we have

$$J(\beta) = J^1_{k,k+1}(x, y, t, t_1, t_2, \dots t_{4k+2}),$$

where

$$\left.\begin{aligned} J^1_{k,k+1}(x, y, t, t_1, t_2, \dots t_{4k+2}) = -\sum_{j=0}^{k-1} \left\{ \int_{t_{4j+4}}^{t_{4j+3}} f(u_1^+(s)\mathrm{d}s \right\} \\ -\sum_{j=0}^{k} \left\{ \int_{t_{4j+2}}^{t_{4j+1}} f(u_2^-(s))\mathrm{d}s \right\} + (t - t_1)f^*\left(\frac{x}{t - t_1}\right) \\ + \sum_{j=1}^{2k} \left\{ (t_{2j} - t_{2j+1}) f^*\left(\frac{1}{t_{2j} - t_{2j+1}}\right) \right\} + t_{4k+2} f^*\left(\frac{-y}{t_{4k+2}}\right). \end{aligned}\right\} \quad (3.4)$$

For $\beta \in \mathscr{C}^1_{k,k}(x, y, t)$, we have

$$J(\beta) = J^1_{k,k}(x, y, t, t_1, t_2, \dots t_{4k}\},$$

where

$$\left.\begin{aligned} J^1_{k,k}(x, y, t, t_1, t_2, \dots t_{4k}) = -\sum_{j=0}^{k-1} \left\{ \int_{t_{4j+4}}^{t_{4j+3}} f(u_1^+(s))\mathrm{d}s \right\} \\ -\sum_{j=0}^{k-1} \left\{ \int_{t_{4j+2}}^{t_{4j+1}} f(u_2^-(s))\mathrm{d}s \right\} + (t - t_1)f^*\left(\frac{x}{t - t_1}\right) \\ + \sum_{j=1}^{2k-1} \left\{ (t_{2j} - t_{2j+1}) f^*\left(\frac{1}{t_{2j} - t_{2j+1}}\right) \right\} + t_{4k} f^*\left(\frac{-y}{t_{4k}}\right). \end{aligned}\right\} \quad (3.5)$$

For $\beta\in\mathscr{C}^0_{k,k}(x,y,t)$, we have

$$J(\beta)=J^0_{k,k}(x,y,t,t_1,t_2,\dots t_{4k}),$$

where

$$\left.\begin{aligned}J^0_{k,k}(x,y,t,t_1,t_2,\dots t_{4k})&=-\sum_{j=0}^{k-1}\left\{\int_{t_{4j+2}}^{t_{4j+1}}f(u_1^+(s))\mathrm{d}s\right\}\\&-\sum_{j=0}^{k-1}\left\{\int_{t_{4j+4}}^{t_{4j+3}}f(u_2^-(s))\mathrm{d}s\right\}+(t-t_1)f^*\left(\frac{x}{t-t_1}\right)\\&+\sum_{j=1}^{2k-1}\left\{(t_{2j}-t_{2j+1})f^*\left(\frac{1}{t_{2j}-t_{2j+1}}\right)\right\}+t_{4k}f^*\left(\frac{-y}{t_{4k}}\right).\end{aligned}\right\}\tag{3.6}$$

For $l=0,1$, $|i-j|\leqslant 1$, $i,j=0,1,2,\dots$, define

$$A^l_{i,j}(x,y,t)=\min_{\beta\in\mathscr{C}^l_{i,j}(x,y,t)}J(\beta).\tag{3.7}$$

From (3.3)–(3.7) it follows that

$$\left.\begin{aligned}A^0_{k+1,k}(x,y,t)&=\min_{\beta\in\mathscr{C}^0_{k+1,k}(x,y,t)}J(\beta)\\&=\min_{0<t_{4k+2}<\dots t_2<t_1\leqslant t}J^0_{k,k}(x,y,t,t_1,t_2,\dots t_{4k+2})\\A^0_{k,k}(x,y,t)&=\min_{\beta\in\mathscr{C}^0_{k,k}(x,y,t)}J(\beta)\\&=\min_{0<t_{4k}<\dots t_2<t_1\leqslant t}J^0_{k,k}(x,y,t,t_1,t_2,\dots t_{4k})\\A^1_{k,k+1}(x,y,t)&=\min_{\beta\in\mathscr{C}^1_{k,k+1}(x,y,t)}J(\beta)\\&=\min_{0<t_{4k+2}<\dots<t_2<t_1\leqslant t}J^1_{k,k+1}(x,y,t,t_1,t_2,\dots,t_{4k})\\A^1_{k,k}(x,y,t)&=\min_{\beta\in\mathscr{C}^1_{k,k}(x,y,t)}J(\beta)\\&=\min_{0<t_{4k}<\dots<t_2<t_1\leqslant t}J^1_{k,k}(x,y,t,t_1,t_2,\dots t_{4k})\end{aligned}\right\}\tag{3.8}$$

It follows from (3.2), (3.7), (3.8) and the definition (2.3) of $Q(x,y,t)$ that

$$\begin{aligned}Q(x,y,t)=\inf_{\{k=0,1,2,\dots\}}&[\min\{A^0_{k,k}(x,y,t),\ A^1_{k,k}(x,y,t),\\&A^0_{k+1,k}(x,y,t),\ A^1_{k,k+1}(x,y,t)\}].\end{aligned}\tag{3.9}$$

Since $u_1^+(s)$ and $u_2^-(s)$ are bounded it follows $Q(x,y,t)$ defined by (2.3) is uniformly bounded in (x,t). Hence it follows from the assumption (A_2) on $f(u)$, that in the minimisation of (3.8) it is enough to consider t_j such that $t_{2j}-t_{2j+1}\geqslant C>0$, where C is a constant depending only on the L^∞ norm of $u_1^+(t)$ and $u_2^-(t)$ and of course on f. The reason for this is that the term $\Sigma(t_{2j}-t_{2j+1})f^*(1/t_{2j}-t_{2j+1})\to\infty$ if at least one of $t_{2j}-t_{2j+1}\to 0$, because of assumption A_2 on f. From this fact the following Lemma immediately follows.

Lemma 3.1. *Let $T>0$ be given, then there exists an integer $N(T)$ depending only on T (and of course on $\|u_1^+(t)\|_\infty$, $\|u_2^-(t)\|_\infty$ and $f(u)$) such that*

$$Q(x,y,t) = \min_{k\in\{0,1,2,\dots N(T)\}} [\min\{A^0_{k,k}(x,y,t),\ A^1_{k,k}(x,y,t),\ A^0_{k+1,k}(x,y,t),\ A^1_{k,k+1}(x,y,t)\}] \tag{3.10}$$

for all $0\leqslant t\leqslant T$, $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$.

Now standard arguments of Conway and Hopf [2] and Lax [5] can be used to show that $A^0_{k+1,k}(x,y,t)$, $A^0_{k,k}(x,y,t)$, $A^1_{k,k+1}(x,y,t)$ and $A^1_{k,k}(x,y,t)$, defined by (3.8) are Lipschitz continuous with respect to (x,y,t). Lemma (3.1) says that $Q(x,y,t)$ is minimum of these Lipschitz continuous functions and hence, we have the following corollary to Lemma (3.1).

COROLLARY 3.2.

$Q(x,y,t)$ is Lipschitz continuous function of (x,y,t).

To proceed further, we need to study more about $A^0_{k+1,k}(x,y,t)$, $A^0_{k,k}(x,y,t)$, $A^1_{k,k+1}(x,y,t)$ and $A^1_{k,k}(x,y,t)$. Let us take the case $A^0_{k+1,k}(x,y,t)$ and the corresponding minimization problem

$$A^0_{k+1,k}(x,y,t) = \min_{t\geqslant t_1>t_2>\dots t_{4k+2}>0} J^0_{k+1,k}(x,y,t,t_1,t_2,\dots t_{4k+2}).$$

Let $(t_1(x,y,t),\ t_2(x,y,t),\dots t_{4k+2}(x,y,t))$ denote a value $(t_1,t_2,\dots t_{4k+2})$ for which minimum is attained. There may be several $(t_1,t_2,\dots t_{4k+2})$ for which this happens. For $j=1,2,\dots 4k+2$, define

$$\left.\begin{aligned} t_j^+(x,y,t) &= \max\{t_j(x,y,t)\} \\ t_j^-(x,y,t) &= \min\{t_j(x,y,t)\}. \end{aligned}\right\}. \tag{3.11}$$

Similar definition can be made for the minimization problem for $J^1_{k,k+1}$, $J^0_{k,k}$ and $J^1_{k,k}$. Let $y_0(x,t)$ denote a value $0\leqslant y_0\leqslant 1$, for which minimum is attained in (2.5), let

$$\left.\begin{aligned} y_0^+(x,t) &= \max\{y_0(x,t)\}, \\ y_0^-(x,t) &= \min\{y_0(x,t)\}. \end{aligned}\right\} \tag{3.12}$$

First we shall prove the following Lemma.

Lemma 3.3. *Let $S_k(x,y,t)$ be any set in $\{\mathscr{C}^0_{k,k}(x,y,t),\ \mathscr{C}^1_{k,k}(x,y,t),\ \mathscr{C}^0_{k+1,k}(x,y,t),\ \mathscr{C}^1_{k,k+1}(x,y,t)\}$. Let β_0 achieve minimum for $\min_{\beta\in S_k(x,y,t)}J(\beta)$. Let (x^*,t^*) be a point on β_0 and β_0^* be the restriction of β_0 on $[0,t^*]$ and let $\beta_0^*\in S_{k_0}(x^*,y,t^*)$ for some k_0, where $S_{k_0}(x^*,y,t^*)$ is one of the sets in $\{\mathscr{C}^0_{kk}(x^*,y,t^*),\ \mathscr{C}^1_{k,k}(x^*,y,t^*),\ \mathscr{C}^0_{k+1,k}(x^*y,t^*),\ \mathscr{C}^1_{k,k+1}(x^*,y,t^*)\}$. Then β_0^* achieves minimum for $\min_{\beta\in S_{k_0}(x^*,y,t^*)}J(\beta)$. The same result is true if one replaces $S_k(x,y,t)$ by $\mathscr{C}(x,y,t)$ and $S_{k_0}(x^*,y,t^*)$ by $\mathscr{C}(x^*,y,t^*)$.*

Proof. Suppose, on the contrary, there exists $\bar\beta\in S_{k_0}(x^*,y,t^*)$ such that

$$J(\beta_0^*) > J(\bar\beta) \tag{3.13}$$

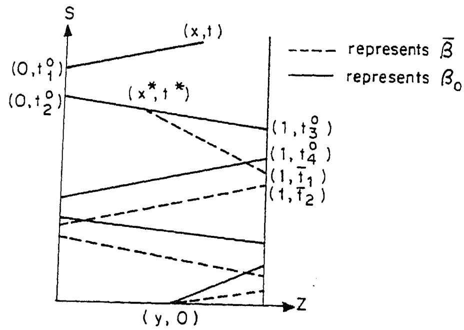

Figure 3.

and $\min_{\beta\in S_{k_0}(x^*,y,t^*)} J(\beta)=J(\bar{\beta})$. There are several cases to consider, among them we take a typical case. The other cases can be treated similarly. We take the case when (x^*,t^*) is in the interior of D and is on the straight line joining $(\beta_0(t_j^0),t_j^0)=(0,t_j^0)$ and $(\beta_0(t_{j+1}^0),t_{j+1}^0)=(1,t_{j+1}^0)$ for some j. Here $t>t_1^0>t_2^0>\ldots>t_j^0>t_{j+1}^0>\ldots>t_{4k+2}^0>0$ corresponds to parameters of β_0 when it meets or leaves the boundary $x=0$ or $x=1$, see (3.8). Thus we assume $\beta_0^*\in S_{k_0}(x^*,y,t^*)=\mathscr{C}^1_{k_0,k_0+1}$, for some k_0. Let $t_1^*>t_2^*>,\ldots>t_{4k_0+2}^*$ be the parameters corresponding to β_0^* and $\bar{t}_1>\bar{t}_2>\ldots\bar{t}_{4k_0+2}$ the parameters corresponding to $\bar{\beta}$. The only interesting case we need to consider is when $t_1^*\neq\bar{t}_1$, see figure 3 for the case $S_k(x,y,t)=\mathscr{C}^0_{2,2}(x,y,t)$, $S_{k_0}(x,y,t)=\mathscr{C}^0_{1,2}(x,y,t)$.

Now define the curve $\beta_1(s)$ on $[0,t)$ by

$$\beta_1(s)=\begin{cases}\beta_0(s) \text{ for } s\in[t^*,t),\\ \bar{\beta}(s) \text{ for } s\in[0,t^*]\end{cases}$$

and

$$\beta_2(s)=\begin{cases}\beta_1 \text{ for } s\in[t_j^0,t]\cup[0,\bar{t}_1]\\ \text{straight line joining } (0,t_j^0) \text{ and } (1,\bar{t}_1) \text{ on } [\bar{t}_1,t_j^0].\end{cases}$$

By Jensen's inequality, we obtain

$$\begin{aligned}\int_{\bar{t}_1}^{t_j^0} f^*\left(\frac{d\beta_1}{ds}\right) &\geq (t_j^0-\bar{t}_1)f^*\left(\frac{1}{t_j^0-\bar{t}_1}\right)\\ &=\int_{\bar{t}_1}^{t_j^0} f^*\left(\frac{d\beta_2}{ds}\right)ds.\end{aligned}\tag{3.14}$$

Now from (3.13) and the definition of $\beta_1(s)$, it follows that

$$J(\beta_0)>J(\beta_1).\tag{3.15}$$

Using (3.14) and the definition of β_2 we obtain from (3.15)

$$J(\beta_0)>J(\beta_1)\geq J(\beta_2).\tag{3.16}$$

By construction $\beta_2\in S_k(x,y,t)$ and hence (3.16) contradicts the fact that β_0 is a minimizer for $\min_{\beta\in S_k(x,y,t)} J(\beta)$. The proof of lemma is complete.

Lemma 3.4. *Let β_1, $i=1,2$ are minimizers for* $\min_{\beta\in\mathscr{C}^0_{k+1,k}(x_i,y,t_i)} J(\beta)$, $t_1 \geqslant t_2$. *Then β_1 and β_2 cannot cross with different slopes in the interior of D. The same result is true when $\mathscr{C}^0_{k+1,k}(x_i,y,t_i)$ is replaced by $\mathscr{C}^1_{k,k+1}(x_i,y,t_i)$ or $\mathscr{C}^0_{k,k}(x_i,y,t_i)$ or $\mathscr{C}^0_{k,k}(x_i,y,t_i)$ or $\mathscr{C}(x_i,y,t_i)$.*

Proof. The proof follows from the previous lemma and a construction similar to the one done in the proof of that Lemma and Jensen's inequality, the details are omitted.

In the next Lemma we obtain some useful properties of $t_1^{\pm}(x,y,t)$.

Lemma 3.5. (i) *Let us consider the minimization problem* $A^0_{k+1,k}(x,y,t) = \min_{0<t_{4k+2}<\dots<t_1} J^0_{k+1,k}(x,y,t,t_1,t_2,\dots t_{4k+2})$ *or* $A^0_{k,k}(x,y,t) = \min_{0<t_{4k}<\dots<t_1} J^0_{k,k}(x,y,t,t_1,t_2,\dots t_{4k})$. *For each fixed $t>0$, $0\leqslant y\leqslant 1$, $t_1^+(.,y,t)$ and $t_1^-(.,y,t)$ are non-increasing function of x. $t_1^+(.,y,t)$ is right continuous and $t_1^-(.,y,t)$ is left continuous. The two functions have the same set of points of discontinuity which is countable subset of $[0,\infty)$ and except on this countable set $t_1^+(.,y,t)$ and $t_1^-(.,y,t)$ are equal. Moreover*

$$\left.\begin{aligned} t_1^+(x,y,t) &= t_1^+(x+0,y,t) = t_1^-(x+0.\,y,t)\,\forall 0\leqslant x<1,\\ t_1^-(x,y,t) &= t_1^-(x-0,y,t) = t_1^+(x-0,y,t)\,\forall 0< x\leqslant 1. \end{aligned}\right\} \tag{3.17}$$

(ii) *Let us consider the minimization problem*

$$A^1_{k,k+1}(x,y,t) = \min_{0<t_{4k+2}<\dots<t_1} J^1_{k,k+1}(x,y,t,t_1,t_2,\dots t_{4k+2})$$

or

$$A^1_{k,k}(x,y,t) = \min_{0<t_{4k}<\dots<t_1} J^1_{k,k}(x,y,t,t_1,t_2,\dots t_{4k}).$$

For each fixed $t\geqslant 0$, $0\leqslant y\leqslant 1$, $t_1^+(.,y,t)$ and $t_1^-(.,y,t)$ are non-decreasing function of x. $t_1^+(.,y,t)$ is left continuous and $t_1^-(.,y,t)$ is right continuous. The two functions have the same set of points of discontinuity which is countable subset of $[0,\infty)$ and except on this countable set $t_1^+(.,y,t)$ and $t_1^-(.,y,t)$ are equal. Moreover

$$\left.\begin{aligned} t_1^+(x,y,t) &= t_1^+(x-0,y,t) = t_1^-(x-0,y,t)\,\forall 0< x\leqslant 1,\\ t_1^-(x,y,t) &= t_1^-(x+0,y,t) = t_1^+(x+0,y,t)\,\forall 0\leqslant x< 1. \end{aligned}\right\} \tag{3.18}$$

(iii) *Let us consider any of the following four minimization problems,*

$$\min_{0<t_{4k+2}<\dots<t_1} J^0_{k+1,k}(x,y,t,t_1,t_2,\dots t_{4k+2}),$$

$$\min_{0<t_{4k}<\dots<t_1} J^0_{k,k}(x,y,t,t_1,t_2,\dots t_{4k}),$$

$$\min_{0<t_{4k+2}<\dots<t_1} J^1_{k,k+1}(x,y,t,t_1,t_2,\dots t_{4k+2}),$$

or

$$\min_{0<t_{4k}<\dots<t_1} J^1_{k,k}(x,y,t,t_1,t_2,\dots t_{4k}).$$

For each fixed $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$, $t_1^+(x,y,.)$ and $t_1(\bar{x},y,.)$ are non-decreasing function of t and $t_1^+(x,y,.)$ is left continuous and $t_1^-(x,y,.)$ is right continuous. The two functions

have the same set of points of discontinuity which is countable subset of $[0,\infty)$ *and except on this countable set* $t_1^+(x,y,.)$ *and* $t_1^-(x,y,.)$ *are equal. Moreover*

$$\left.\begin{aligned} t_1^+(x,y,t) = t_1^+(x,y,t-0) = t_1^-(x,y,t-0)\forall t>0,\\ t_1^-(x,y,t) = t_1^-(x,y,t+0) = t_1^+(x,y,t+0)\forall t>0.\end{aligned}\right\} \tag{3.19}$$

(iv) *For each fixed* $0 \leqslant y \leqslant 1$, $t_1^+(x,y,t) = t_1^-(x,y,t)$ *a.e.* (x,t).

Proof. We shall prove (i). From the definition of $t_1^\pm(x,y,t)$ and using the fact that two minimizers cannot cross, in the interior of D, see Lemma 3.4, we get if $x_1 < x_2$

$$t_1^-(x_2,y,t) \leqslant t_1^+(x_2,y,t) \leqslant t_1^-(x_1,y,t) \leqslant t_1^+(x_1,y,t). \tag{3.20}$$

This inequality shows that $t_1^-(.,y,t)$ is non-increasing and hence $t_1^-(.,y,t)$ have atmost a countable number of discontinuity points. By the continuity of $A_{k+1,k}^0$, (3.20) implies (3.17) also.

Proofs of (ii) and (iii) are similar and (iv) follows from (i), (ii) and (iii). Proof of the Lemma is complete.

Now let us compute the left and right derivatives of $A_{k,k}^0(x,y,t)$, $A_{k+1,k}^0(x,y,t)$, $A_{k,k}^1(x,y,t)$ and $A_{k,k+1}^1(x,y,t)$ with respect to x and t, for each fixed $0 \leqslant y \leqslant 1$. Denote

$$\frac{\partial A^\pm}{\partial x} = \lim_{\substack{h\to 0\\ h>0}} \left[\frac{A(x\pm h,y,t)-A(x,y,t)}{h}\right]$$

$$\frac{\partial A^\pm}{\partial t}(x,y,t) = \lim_{\substack{h\to 0\\ h>0}} \left[\frac{A(x,y,t\pm h)-A(x,y,t)}{h}\right]$$

We shall prove the following Lemma.

Lemma 3.6. *For* $k = 1,2,\ldots$

(i) $$\frac{\partial (A_{k,k}^0)^+}{\partial x}(x,y,t) = (f^*)'\left(\frac{x}{t-t_1^+(x,y,t)}\right)\forall 0\leqslant x<1,$$

(ii) $$\frac{\partial (A_{k,k}^0)^-}{\partial x}(x,y,t) = (f^*)'\left(\frac{x}{t-t_1^-(x,y,t)}\right)\forall 0< x\leqslant 1,$$

(iii) $$\frac{\partial (A_{k+1,k}^0)^+}{\partial x}(x,y,t) = (f^*)'\left(\frac{x}{t-t_1^+(x,y,t)}\right)\forall 0\leqslant x<1,$$

(iv) $$\frac{\partial (A_{k+1,k}^0)^-}{\partial x}(x,y,t) = (f^*)'\left(\frac{x}{t-t_1^-(x,y,t)}\right)\forall 0< x\leqslant 1,$$

(v) $$\frac{\partial (A_{k,k}^1)^+}{\partial x}(x,y,t) = (f^*)'\left(\frac{x}{t-t_1^-(x,y,t)}\right)\forall 0\leqslant x<1,$$

(vi) $$\frac{\partial (A_{k,k}^1)^-}{\partial x}(x,y,t) = (f^*)'\left(\frac{x}{t-t_1^+(x,y,t)}\right)\forall 0< x\leqslant 1,$$

(vii) $$\frac{\partial (A_{k,k+1}^1)^+}{\partial x}(x,y,t) = (f^*)'\left(\frac{x}{t-t_1^-(x,y,t)}\right)\forall 0\leqslant x<1,$$

(viii) $\dfrac{\partial (A^1_{k,k+1})^-}{\partial x}(x,y,t) = (f^*)'\left(\dfrac{x}{t - t_1^+(x,y,t)}\right) \forall 0 < x \leqslant 1,$

(ix) $\dfrac{\partial (A^0_{k,k})^\pm}{\partial t}(x,y,t) = -f\left[(f^*)'\left(\dfrac{t}{t - t_1^\mp(x,y,t)}\right)\right] \forall t > 0,$

(x) $\dfrac{\partial (A^0_{k+1,k})^\pm}{\partial t}(x,y,t) = -f\left[(f^*)'\left(\dfrac{t}{t - t_1^\mp(x,y,t)}\right)\right] \forall t > 0,$

(xi) $\dfrac{\partial (A^1_{k,k})^\pm}{\partial t}(x,y,t) = -f\left[(f^*)'\left(\dfrac{t}{t - t_1^\mp(x,y,t)}\right)\right] \forall t > 0,$

(xii) $\dfrac{\partial (A^1_{k+1,k})^\pm}{\partial t}(x,y,t) = -f\left[(f^*)'\left(\dfrac{t}{t - t_1^\mp(x,y,t)}\right)\right] \forall t > 0,$

Proof. We shall prove (i). By the definition of $A^0_{k,k}(x,y,t)$, we have

$$\begin{aligned}
A^0_{k,k}(x+h,y,t) \leqslant &- \sum_{j=0}^{k-1} \int_{t_{4j+2}}^{t_{4j+1}(x,y,t)} f(u_1^+(s))\mathrm{d}s - \sum_{j=0}^{k-1} \int_{t_{4j+4}}^{t_{4j+3}(x,y,t)} f(u_2^-(s))\mathrm{d}s \\
&+ t_{4k}(x,y,t) f^*\left(\frac{-y}{t_{4k}(x,y,t)}\right) \\
&+ \sum_{j=1}^{2k-1} t_{2j}(x,y,t) - t_{2j+1}(x,y,t) f^*\left(\frac{1}{t_{2j}(x,y,t) - t_{2j+1}(x,y,t)}\right) \\
&+ (t - t_1^+(x,y,t)) f^*\left(\frac{x+h}{t - t_1^+(x,y,t)}\right)
\end{aligned}$$

and

$$\begin{aligned}
A^0_{k,k}(x,y,t) = &- \sum_{j=0}^{k-1} \int_{t_{4j+2}}^{t_{4j+1}(x,y,t)} f(u_1^+(s))\mathrm{d}s - \sum_{j=0}^{k-1} \int_{t_{4j+4}}^{t_{4j+3}(x,y,t)} f(u_2^-(s))\mathrm{d}s \\
&+ t_{4k}(x,y,t) f^*\left(\frac{-y}{t_{4k}(x,y,t)}\right) \\
&+ \sum_{j=1}^{2k-1} (t_{2j}(x,y,t) - t_{2j+1}(x,y,t)) f^*\left(\frac{1}{t_{2j}(x,y,t) - t_{2j+1}(x,y,t)}\right) \\
&+ (t - t_1^+(x,y,t)) f^*\left(\frac{x}{t - t_1^+(x,y,t)}\right)
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{A^0_{k,k}(x+h,y,t) - A^0_{k,k}(x,y,t)}{h} \\
&\leqslant (t - t_1^+(x,y,t))\left[f^*\left(\frac{x+h}{t - t_1^+(x,y,t)}\right) - f^*\left(\frac{x}{t - t_1^+(x,y,t)}\right)\right] \\
&\leqslant (f^*)'\left(\frac{x+\eta(h)}{t - t_1^+(x,y,t)}\right), \ 0 < \eta(h) < h.
\end{aligned}$$

Letting $h \to 0$, we get

$$\frac{\partial A^0_{k,k}}{\partial x}(x,y,t) \leqslant (f^*)'\left(\frac{x}{t - t_1^+(x,y,t)}\right). \tag{3.21}$$

In a similar way, we get

$$\frac{A^0_{k,k}(x+h,y,t) - A^0_{k,k}(x,y,t)}{h} \geqslant (f^*)'\left(\frac{x+\eta(h)}{t - t_1^+(x+h,y,t)}\right),\ 0<\eta(h)<h.$$

Letting $h \to 0$ and using right continuity of $t_1^+(.,y,t)$ we get

$$\frac{\partial (A^0_{k,k})^+}{\partial x}(x,y,t) \geqslant (f^*)'\left(\frac{x}{t - t_1^+(x,y,t)}\right). \tag{3.22}$$

From (3.21) and (3.22) we get (i). The proof of (i) is complete. Similar argument can be used to prove (ii)–(viii). The details are omitted.

Next we compute the right derivative of $A^0_{k,k}(x,y,t)$ with respect to t. As before

$$\frac{A^0_{k,k}(x,y,t+h) - A^0_{k,k}(x,y,t)}{h}$$

$$\leqslant \frac{(t+h-t_1^-(x,y,t))f^*\left(\frac{x}{t+h-t_1^-(x,y,t}\right) - (t-t_1^-(x,y,t))f^*\left(\frac{x}{t-t_1^-(x,y,t)}\right)}{h}.$$

Letting $h \to 0$, we get

$$\frac{\partial (A^0_{k,k})^+}{\partial t}(x,y,t) \leqslant f^*\left(\frac{x}{t-t_1^-(x,y,t)}\right) - \left(\frac{x}{t-t_1^-(x,y,t)}\right)(f^*)'\left(\frac{x}{t-t_1^-(x,y,t)}\right).$$

As in (3.22), using the right continuity of $t_1^-(.,y,t)$ we can show

$$\frac{\partial (A^0_{k,k})^+}{\partial t}(x,y,t) \geqslant f^*\left(\frac{x}{t-t_1^-(x,y,t)}\right) - \left(\frac{x}{t-t_1^-(x,y,t)}\right)(f^*)'\left(\frac{x}{t-t_1^-(x,y,t)}\right)$$

and hence

$$\begin{aligned}\frac{\partial (A^0_{k,k})^+}{\partial t}(x,y,t) = {} & f^*\left(\frac{x}{t-t_1^-(x,y,t)}\right) \\ & - \left(\frac{x}{t-t_1^-(x,y,t)}\right)(f^*)'\left(\frac{x}{t-\bar{t}_1(x,y,t)}\right).\end{aligned} \tag{3.23}$$

For convex functions, the following identity is true, namely,

$$f[(f^*)'(s)] = s(f^*)'(s) - f^*(s).$$

Using this in (3.23) we get

$$\frac{\partial (A^0_{k,k})^+}{\partial t}(x,y,t) = -f\left[(f^*)'\left(\frac{x}{t-\bar{t}_1(x,y,t)}\right)\right].$$

The same method can be used to prove (x) – (xii). The proof of Lemma is complete.
Next we shall prove the following Lemma.

Lemma 3.7. *For each fixed* $y > 0$,

$$Q_t + f(Q_x) = 0 \quad \text{a.e } (x, t).$$

Proof. By definition

$$A^0_{0,0}(x, y, t) = t\, f^*\left(\frac{x-y}{t}\right)$$

and hence

$$\frac{\partial(A^0_{0,0})}{\partial x}(x, y, t) = (f^*)'\left(\frac{x-y}{t}\right)$$

and

$$\frac{\partial(A^0_{0,0})}{\partial t}(x, y, t) = f^*\left(\frac{x-y}{t}\right) - (f^*)'\left(\frac{x-y}{t}\right)\cdot\left(\frac{x-y}{t}\right)$$
$$= -f(f^*)'\left(\frac{x-y}{t}\right).$$

From these, it follows that $A^0_{0,0}(x, y, t)$ satisfies

$$\frac{\partial v}{\partial t} + f\left(\frac{\partial v}{\partial x}\right) = 0 \tag{3.24}$$

for each fixed $0 \leqslant y \leqslant 1$. It follows from Lemma (3.6) that $A^0_{k,k}(x, y, t)$, $A^1_{k,k}(x, y, t)$, $A^1_{k,k+1}(x, y, t)$ and $A^0_{k+1,k}(x, y, t)$ satisfies (3.24) a.e. (x,t), for each fixed $0 \leqslant y \leqslant 1$. Also for each fixed T, $Q(x, y, t)$ is the minimum of a finite number of functions which satisfies (3.24), in $0 < x < 1$, $0 < t < T$, see Lemma (3.2). Now recall the following result of Conway and Hopf [2]: If $\{v^i(x, t){:}\ i = 1, 2, .. N\}$ solves (3.24), so does $v(x, t)$ defined by

$$v(x, t) = \min_{i=1,2,..N} v^i(x, t).$$

Using this fact we get, for fixed $0 \leqslant y \leqslant 1$

$$Q_t + f(Q_x) = 0 \quad \text{a.e. } (x, t),\ 0 < x < 1,\ 0 < t < T.$$

But since T is arbitrary, Lemma follows.
Let (x, y, t) be fixed and let $\beta \in \mathscr{C}(x, y, t)$. Define

$$H(x, y, t, \beta) = \int_0^y u_0(z) + J(\beta), \tag{3.25}$$

then (2.5) can be rewritten in the following way.
Let $\bar{\beta} \in \mathscr{C}(x, y, t)$ and $y_0(x, t)$ be such that

$$\min_{\substack{\beta \in \mathscr{C}(x, y, t)\\ y \geqslant 0}} [H(x, y, t, \beta)] = H(x, y_0(x, t), t, \bar{\beta})). \tag{3.25}$$

Note that RHS of (3.25) is nothing but

$$\int_0^{y_0(x,t)} u_0(z) + Q(x, y_0(x,t), t),$$

we call it $U(x,t)$ i.e.,

$$U(x,t) = \int_0^{y_0(x,t)} u_0(z) + Q(x, y_0(x,t), t).$$

Let $y_0^+(x,t)$ and $y_0^-(x,t)$ be defined by (3.12). We have the following lemma.

Lemma 3.8. *Let $t > 0$, be fixed,*

(i) *$y_0^+(x,t)$ and $y_0^-(x,t)$ are non-decreasing function of x, $y_0^+(x,t)$ is right contiuous and $y_0^-(x,t)$ is left continuous. The two functions have the same set of point of discontinuity and except at these countably many points, the two functions are equal. Moreover,*

$$\left.\begin{aligned} y_0^+(x,t) &= y_0^+(x+0,t) = y_0^-(x+0,t),\\ y_0^-(x,t) &= y_0^-(x-0,t) = y_0^+(x-0,t). \end{aligned}\right\} \tag{3.26}$$

(ii) *Suppose the minimum in* (3.25) *for $H(\bar{x}, y, t, \beta)$ is attained for some $\bar{\beta} \in \mathscr{C}^0_{k,k}(\bar{x}, y_0\cdot(\bar{x},t),t)$ $(\mathscr{C}^0_{k+1,k}(\bar{x}, y_0(\bar{x},t),t))$. Let $x^* < \bar{x}$, and let β^* attain minimum for $H(x^*, y, t, \beta)$ then $\beta^* \in \mathscr{C}^0_{k,k}(x^*, y_0(x^*,t)t)$. Moreover, for $0 \leqslant x^* < \bar{x}$*

$$\begin{aligned} &t_j^{\pm}(\bar{x}, y_0^{\pm}(\bar{x},t), t) = t_j^{\pm}(x^*, y_0^{\pm}(x^*,t),t),\ j \geqslant 2,\\ &y_0^-(\bar{x},t) = y_0^+(x^*,t) = y_0^-(x^*,t). \end{aligned} \tag{3.27}$$

(iii) *Suppose the minimum in* (3.25) *for $H(\bar{x}, y, t, \beta)$ is attained for some $\bar{\beta} \in \mathscr{C}^1_{k,k}(\bar{x}, y_0\cdot(\bar{x},t),t)$ $(\mathscr{C}^1_{k,k+1}(x, y_0(\bar{x},t),t))$ let $x^* > \bar{x}$, and let β^* attain minimum in* (3.25) *for $H(x^*, y, t, \beta)$ then $\beta^* \in \mathscr{C}^1_{k,k}(x^*, y_0(x,t),t)$ $(\mathscr{C}^1_{k,k+1}(x^*, y_0(x^*,t),t))$. Moreover, $\forall \bar{x} < x^* \leqslant 1$*

$$\begin{aligned} &t_j^{\pm}(\bar{x}, y_0^+(\bar{x},t),t) = t_j^{\pm}(x^*, y_0^{\pm}(x^*,t,t)),\ j \geqslant 2\\ &y_1^+(\bar{x},t) = y_0^+(x^*,t) = y_0^-(x^*,t). \end{aligned} \tag{3.28}$$

Proof. Proof of (i) is exactly the same as the proof of Lemma (3.5). We shall prove (ii). Let us take the case where $\bar{\beta} \in \mathscr{C}^0_{k,k}(\bar{x}, y_0(\bar{x},t),t)$, $k \geqslant 1$. Since $\bar{\beta}$ and β^* cannot interest in the interior of D, see Lemma (3.4), it follows that $\beta^* \in \mathscr{C}^0_{k,k}(x^*, y_0(x^*,t),t)$ and

$$t_j^{\pm}(\bar{x}, y_0^{\pm}(\bar{x},t),t) = t_j^{\pm}(x^*, y_0^{\pm}(x^*,t),t)\ j \geqslant 2$$

and

$$\bar{y}_0(\bar{x},t) = y_0^-(x^*,t).$$

Now using part (i), we get $y^+(x^*,t) = y^{-1}(x^*,t)$. Proof of (iii) is similar. This completeness the proof of lemma.

Now we shall prove the main theorem.

Proof of main theorem. Following Lax [5], we introduce

$$\left.\begin{aligned}
u_N(x,t) &= \frac{\displaystyle\int_0^1 Q_1(x,y,t)\exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y}{\displaystyle\int_0^1 \exp\left\{-N\left[\int_0^y u_0(z) + Q(x,y,t)\right]\right\}\mathrm{d}y}\\
f_N(x,t) &= \frac{\displaystyle\int_0^1 f(Q_1(x,y,t))\exp\left\{-N\left[\int_0^y u(z) + Q(x,y,t)\right]\right\}\mathrm{d}y}{\displaystyle\int_0^1 \exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y}\\
V_N(x,t) &= \int_0^1 \exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y\\
U_N(x,t) &= \frac{1}{N}\log V_N.
\end{aligned}\right\} \tag{3.29}$$

As in Lax [5], it follows that,

$$\lim_{N\to\infty} u_N(x,t) = Q_1(x, y_0(x,t), t)$$

$$\lim_{N\to\infty} f_N(x,t) = f(Q_1(x, y_0(x,t), t))$$

and

$$\lim_{N\to\infty} U_N(x,t) = U(x,t) = \int_0^{y_0(x,t)} u_0(z)\mathrm{d}z + Q(x, y_0(x,t), t) \tag{3.30}$$

where $y_0(x,t)$ minimizes (2.5).

Also

$$u_N(x,t) = -\frac{1}{N}\frac{(V_N)_x}{V_N} = (U_N)_x. \tag{3.31}$$

It follows from (3.29) and (3.30) that

$$\frac{\partial U}{\partial x}(x,t) = Q_1(x, y_0(x,t), t).$$

Next we shall show that

$$(U_N)_t = -f_N. \tag{3.32}$$

We consider

$$(U_N)_t = -\frac{1}{N}\frac{(V_N)_t}{V_N}$$

$$= \frac{\displaystyle\int_0^1 \frac{\partial Q}{\partial t}(x,y,t)\exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y}{\displaystyle\int_0^1 \exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y}$$

$$= \frac{\displaystyle\int_0^1 f(Q_1(x,y,t))\exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y}{\displaystyle\int_0^1 \exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y}$$

To obtain the last equality we used Lemma (3.7) and Lemma (3.8). Now (3.32) follows from the definition of f_N. From (3.31) and (3.32) we get

$$(u_N)_t + (f_N)_x = 0.$$

Hence for all test functions $\varphi(x,t)\in C_0^\infty[(0,\infty)\times(0,\infty)]$ we get

$$\int_0^\infty\int_0^1 (u_N\varphi_t + f_N\varphi_x)\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.33}$$

Let $N\to\infty$, in (3.33) and use (3.30) to get

$$\int_0^\infty\int_0^1 (u\varphi_t + f(u)\varphi_x)\mathrm{d}x\,\mathrm{d}t = 0.$$

Now we shall show that $u(x,t)$ satisfies the initial condition. By the argument similar to be Lemma (3.1) we get given $\varepsilon>0$, $\exists\delta>0$ such that for all $\varepsilon\leqslant x\leqslant 1-\varepsilon$, $t\leqslant\delta$,

$$u(x,t) = (f^*)'\left(\frac{x-y_0(x,t)}{t}\right)$$

where $y_0(x,t)$ minimizes

$$\min_{0\leqslant y\leqslant 1}\left[\int_0^y u_0(z)\mathrm{d}z + tf^*\left(\frac{x-y}{t}\right)\right].$$

But then Lax's argument [5] can be used to show

$$\lim_{t\to 0} u(x,t) = u_0(x) \quad \text{a.e. } \varepsilon\leqslant x\leqslant 1-\varepsilon.$$

Since $\varepsilon>0$, is arbitrary, it follows that,

$$\lim_{t\to 0} u(x,t) = u_0(x) \quad \text{a.e. } 0\leqslant x\leqslant 1.$$

Next we show that $u(x,t)$ satisfy the entropy condition (1.5). Because of Lemmas (3.5), (3.6) and (3.8) and the definition of Q_1, Lemma (3.6) it is clear that $u(x\pm 0,t)$ exists for all $0<x<1$. In fact

$$u(x\pm 0,t)=Q_1(x,y_0^{\pm}(x,t),\ t).$$

Now the entropy condition (1.5) follows from the definition of Q_1, Lemma (3.6) and the increasing nature of $(f^*)'$.

Lastly we show that $u(x,t)$ satisfies the boundary condition $(1.3)'_0$ and $(1.3)'_1$. Here again the existence of $u(0+,t)$ and $u(1-,t)$ follows as before. We shall verify that $u(0+,t)$ satisfies $(1.3)'_0$. The proof of $(1.3)'_1$ is similar and hence omitted.

To verify $(1.3)'_0$, at $t=t_0$, first notice that $u(x,t)$ defined by (2.6) satisfies the semigroup property i.e.

$$u(x,t_0)=\bar{u}(x,t_0)$$

where $\bar{u}(x,t)$ is defined by (2.6) with initial data is prescribed at time t_1, $0<t_1<t_0$:

$$\bar{u}(x,t_1)=u(x,t_1).$$

Because of this semigroup property and the condition (A_2), an argument similar to the proof of Lemma (3.1) can be used to show that

$$u(x,t)=Q_1(x,y_0(x,t),t)\forall t_0-\varepsilon\leqslant t\leqslant t_0,$$

for some $\varepsilon>0$ sufficiently small. Here $y_0(x,t)$ minimizes

$$\min_{0\leqslant y\leqslant 1}\left[\int_0^y u(z,t_0-\varepsilon)dz+Q^0(x,y,t)\right]$$

and

$$Q^0(x,y,t)=\min\{A^0_{1,0}(x,y,t),A^0_{0,0}(x,y,t),A^0_{0,1}(x,y,t)\}$$

where

$$A^0_{1,0}(x,y,t)=\min_{t_0-\varepsilon<t_2<t_1<t_0}\left[-\int_{t_2}^{t_1}f(u_1^+(s))ds+(t_0-t_1)f^*\left(\frac{x}{t_0-t_1}\right)\right.$$
$$\left.+(t_2-t_0+\varepsilon)f^*\left(\frac{-y}{t_2-t_0+\varepsilon}\right)\right]$$

$$A^0_{0,0}(x,y,t)=(t_0-(t_0-\varepsilon))f^*\left(\frac{x-y}{t_0-t_1+\varepsilon}\right)=\varepsilon f^*\left(\frac{x-y}{\varepsilon}\right)$$

$$A^0_{0,1}(x,y,t)=\min_{t_0-\varepsilon<t_2<t_1<t_0}\left[-\int_{t_2}^{t_1}f(u_2^-(s)ds+(t_0-t_1)f^*\left(\frac{x-1}{t_0-t_1}\right)\right.$$
$$\left.+(t_2-t_0+\varepsilon)f^*\left(\frac{-y}{t_2-t_0+\varepsilon}\right)\right].$$

Again because of the condition (A_2) it follows that if $x\leqslant\delta$, $(t_0-t_1)f^*(x-1/t_0-t_1)\to\infty$ as $t_0-t_1\to 0$. Hence for $\varepsilon>0$ sufficiently small one has

$$u(x,t)=Q_1(x,y_0(x,t),t)\forall t_0-\varepsilon<t\leqslant t_0,\ 0\leqslant x<\delta$$

where $y_0(x,t)$ minimizes

$$\min_{0 \leqslant y \leqslant 1} \left[\int_0^y u(z, t_0 - \varepsilon) dz + Q^0(x, y, t) \right]$$

and

$$Q^0(x, y, t) = \min[A^0_{1,0}(x, y, t), A^0_{0,0}(x, y, t)].$$

In this case $u(x,t)$ satisfies $(1.3)'_0$ was proved in [4]. The proof of theorem is complete.

4. Weighted Burger's Equation

Equations of the type

$$(x^\alpha u)_t + \left(x^\alpha \frac{u^2}{2} \right)_x = 0, \ \alpha > -2 \tag{4.1}$$

are interesting, because such kind of equations appear in fluid dynamics with spherical and cylindrical symmetry and is studied by Lefloch [3] in the quarter plane $x > 0$, $t > 0$.

As is observed in [3], a change of variable

$$v(y, t) = \left(\frac{\alpha}{2} + 1 \right) x^{\alpha/2} u(x, t), \ y = x^{\alpha/2 + 1} \tag{4.2}$$

transforms (4.1) into the Burgers equation.

$$v_t + \left(\frac{v^2}{2} \right)_x = 0. \tag{4.3}$$

Thus $u(x,t)$ is a solution of (4.1) iff v is a solution of (4.3). From the Bardos *et al* [1] formulation of the initial boundary value problem one easily gets the following formulation of the initial and boundary condition for (4.1) in $D = \{(x,t): 0 \leqslant x \leqslant 1, t > 0\}$.

Initial data for (4.1):

$$u(x, 0) = u_0(x) \ 0 \leqslant x \leqslant 1. \tag{4.4}$$

Boundary condition at $x = 0$.

$$\left.\begin{array}{l} \lim\limits_{x \to 0} [x^{\alpha/2} u(x,t)] = u_1^+(t) \\ \text{or} \\ \lim\limits_{x \to 0} [x^{\alpha/2} u(x,t)] \leqslant 0 \text{ and } \lim\limits_{x \to 0} x^\alpha u^2(0 + t_1) \geqslant u_1^+(t)^2 \end{array}\right\} \text{a.e. } t > 0. \tag*{$(4.5)_0$}$$

Boundary condition at $x = 1$:

$$\left.\begin{array}{l} u(1-, t) = u_2^-(t) \\ \text{or} \\ u(1-, t) \geqslant 0 \text{ and } u^2(1-, t) \geqslant u_2^-(t)^2 \end{array}\right\} \text{a.e. } t > 0. \tag*{$(4.5)_1$}$$

Here we used the notation $u_1^+(t) = \max(u_1(t), 0)$, $u_2^-(t) = \min(u_2(t), 0)$.

Entropy condition:

$$u(x+, t) \leqslant u(x-t) \tag{4.6}$$

From the main theorem we get the following explicit formula for $u(x, t)$, the solution of the problem (4.1), (4.4) $(4.5)_0$, $(4.5)_1$ and (4.6).

$$u(x, t) = \left(\frac{2}{\alpha+2}\right) x^{-\alpha/2} v(x^{\alpha+2/2}, t)$$

where $v(y, t)$ is given by

$$v(y, t) = Q_1(y, z_0(y, t), t), \tag{4.7}$$

and $z_0(y, t)$ minimizes

$$\min_{\substack{\beta \in \mathscr{C}(y,z,t) \\ 1 \geqslant z \geqslant 0}} \left[\int_0^z \left(\frac{\alpha}{2}+1\right) z_1^{\alpha/\alpha+2} u_0(z_1^{2/\alpha+2}) \mathrm{d}z_1 - \frac{(\alpha+2)^2}{8} \int_{\{s:\beta(s)=0\}} (u_1^+(s))^2 \mathrm{d}s \right.$$
$$\left. \frac{(\alpha+2)^2}{8} \int_{\{s:\beta(s)=1\}} (u_2^-(s))^2 \mathrm{d}s + \frac{1}{2} \int_{\{s:0<\beta<1\}} \left(\frac{\mathrm{d}\beta}{\mathrm{d}s}\right)^2 \mathrm{d}s \right].$$

In (4.7), $Q_1(y, z_0(y, t), t)$ is defined by (2.2), (2.3) and (2.4) with $f(u) = u^2/2$.

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An example of a regular space that is not completely regular

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Abstract. A simpler example of regular space that is not completely regular is attempted.

Keyword. Regular space.

1. Introduction

In 1925 Urysohn [10] posed, but left unanswered, the question of whether or not regular topological spaces exist in which every continuous real-valued function is constant. Tychonoff [9], in an attempt to settle this question, produced an example of a regular space which is not completely regular. Later, making essential use of the example of Tychonoff, Hewitt [5], Novák [7], van Est-Freudenthal [3] and Herrlich [4] constructed regular spaces supporting no non-constant continuous real-valued function. Among the earliest treatises on set topology, Čech [1] gives an account of Novák's example and Vaidyanathaswamy [11] presents Tychonoff's example mentioned above. In recent times, more accessible references are Dugundji [2] and Steen and Seebach [8] which give the same examples under the names "Spiral staircase" or "Tychonoff corkscrew". This example involves the use of the uncountable well-ordered space ω_1. I venture, in this shortnote, on an apparently simpler construction of a regular space that is not completely regular.

2. Construction

For any even integer n let $T_n = \{n\} \times (-1, 1)$ and $X_1 = \cup_{n\,\text{even}} T_n = \{(n, y): n \text{ even integer}, -1 < y < 1\}$.

Let $\{\alpha_k, k \geqslant 1\}$ be a strictly increasing sequence of positive real numbers such that $\lim_{k\to\infty} \alpha_k = 1$.

For any odd integer n, set $C_{n,k} = \{(x, y): (x-n)^2 + y^2 = \alpha_k^2\}$, $k = 1, 2, \ldots$ and set $X_2 = \cup_{n\,\text{odd}} \cup_{k=1}^{\infty} C_{n,k}$. Let a and b be two distinct points not belonging to the union $X_1 \cup X_2$. Form the set $X = X_1 \cup X_2 \cup \{a, b\}$.

Topology of X

We shall define a topology on X by describing the neighbourhoods of each of its points.

For each odd integer n and each $k \geqslant 1$, all points of $C_{n,k}$ except the point $a_{n,k} = (n, \alpha_k)$

are isolated. A neighbourhood of $a_{n,k}$ consists of all but a finite number of points of $C_{n,k}$. Write

$$C_n = \bigcup_{k=1}^{\infty} C_{n,k}, \quad n \text{ odd.}$$

If $p=(n, y)\in X_1$, consider the subset

$$\{(z, y): n-1 < z < n+1\} \cap (C_{n-1} \cup C_{n+1})$$

of X. A neighbourhood of p consists of all but a finite number of points of this subset. A neighbourhood of a consists of all points of $X_1 \cup X_2$ with first coordinate greater than some real number c. A neighbourhood of b consists of points of $X_1 \cup X_2$ with first coordinate less than some real number d. The neighbourhoods describe a T_1 topology on X. It is not difficult to see that under this topology each neighbourhood of a point of X contains a closed neighbourhood of the same point. X is thus regular and Hausdorff.

Failure of complete regularity

We claim that given a real-valued, continuous function f on X, $f(a)=f(b)$. Consequently X fails to be completely regular.

Let us first observe that if h is a continuous real-valued function on $C_{n,k}$, the set

$$\{(x, y)\in C_{n,k}: h(x, y) \neq h(a_{n,k})\}$$
$$= \left\{(x, y)\in C_{n,k}: h(x, y) \notin \bigcap_{m=1}^{\infty} \left(h(a_{n,k}) - \frac{1}{m}, h(a_{n,k}) + \frac{1}{m}\right)\right\}$$

is at most a countable subset of $C_{n,k}$.

Let $f: X \to \mathbb{R}$ be an arbitrary, continuous function. Set $B_{n,k} = \{(x, y)\in C_{n,k}: f(x, y) \neq f(a_{n,k})\}$ and $D_n =$ ordinates of points in $\cup_{k=1}^{\infty} B_{n,k}\}$. In view of the observation above, each $B_{n,k}$ is countable and consequently, each D_n is so. If $D = \cup_{n \text{odd}} D_n$, D is then a countable subset of $(-1, 1)$. Suppose $p\in X_1$ is such that $p\in T_n$ and the ordinate y of p does not belong to D. Consider

$$\{(z, y): n-1 < z < n+1\} \cap (C_{n-1} \cup C_{n+1}).$$

If

$$(z, y)\in C_{n-1,k}, f(z, y) = f(a_{n-1,k})$$

and if

$$(z, y)\in C_{n+1,k}, f(z, y) = f(z, y) = f(a_{n+1,k}).$$

From the structure of neighbourhoods of p it is clear that

$$f(p) = \lim_{k\to\infty} f(a_{n-1,k}) = \lim_{k\to\infty} f(a_{n+1,k}).$$

Let $q \in X_1$ be such that $q \in T_{n+2}$ and the ordinate of q does not belong to D. Considerations as above will lead us to conclude that

$$f(q) = \lim_{k \to \infty} f(a_{n+3,k}) = \lim_{k \to \infty} f(a_{n+1,k}).$$

Hence, $f(p) = f(q)$.

If $G = \{(n, y) : n \text{ even and } y \in (-1, 1) - D\}$, the above argument shows that for any $p \in G$, $f(p) = \lim_{k \to \infty} f(a_{n+1,k}) = \lim_{k \to \infty} f(a_{n-1,k})$ where $p = (n, y)$. Thus f is a constant on G, say, α. Since f assumes the value α in every neighbourhood of each of a and b, $f(a) = \alpha = f(b)$. The claim is thus established.

A few remarks about the space X

(I) The space X has the merit of playing the role of the space Q which enters into the construction, due to Herrlich [4, page 153], of a regular space on which every continuous real-valued function is constant.

(II) The space X admits a proper subspace which is also a regular Hausdorff space that fails to be completely regular. To be precise take the subspace Z of X where

$$Z = \left(\bigcup_{\substack{n \geqslant 0 \\ n \text{ even}}} T_n \right) \bigcup \left(\bigcup_{\substack{n \geqslant 1 \\ n \text{ odd}}} \bigcup_{k=1}^{\infty} C_{n,k} \right) \bigcup \{a\}.$$

T_0 and $\{a\}$ are disjoint closed subsets of Z. If $g : Z \to \mathbb{R}$ is a continuous function which is 1 on T_0, it can be easily seen that $g(a) = 1$. As a result, Z cannot be completely regular.

(III) At the time of the construction of the space X, the author was not aware of the existence of the paper by Mysior [6] which contains an elementary example of regular space which is not completely regular. However the space X is a different example.

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Factors for $|\bar{N}, p_n; \delta|_k$ summability of Fourier series

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Abstract. In this paper two theorems on $|\bar{N}, p_n; \delta|_k$ summability factors, which generalize the results of Bor [4] on $|\bar{N}, p_n|_k$ summability factors, have been proved.

Keywords. Fourier series; summability factors; absolute summability.

1. Introduction

Let Σa_n be a given infinite series with partial sums (s_n) and let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geqslant 1). \tag{1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n). The series Σa_n is said to be summable $|\bar{N}, p_n|_k, k \geqslant 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |t_n - t_{n-1}|^k < \infty \tag{3}$$

and it is said to be summable $|\bar{N}, p_n; \delta|_k, k \geqslant 1$ and $\delta \geqslant 0$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1} |t_n - t_{n-1}|^k < \infty. \tag{4}$$

In the special case when $p_n = 1$ for all values of n (resp. $\delta = 0$), $|\bar{N}, p_n; \delta|_k$ summability is the same as $|C, 1; \delta|_k$ (resp. $|\bar{N}, p_n|_k$) summability.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t) \mathrm{d}t = 0 \tag{5}$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{6}$$

We write

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}, \quad \varphi_1(t) = \frac{1}{2}\int_0^t \varphi(u)\,du \text{ and } \Delta\lambda_n = \lambda_n - \lambda_{n+1}.$$

2. Quite recently Bor [4] proved the following theorems.

Theorem A. *Let the sequence (p_n) be such that*

$$P_n = O(np_n) \tag{7}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{8}$$

If $\varphi_1(t)$ is of bounded variation in $(0,\pi)$ and (λ_n) is a sequence such that

$$\sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n|^k < \infty \tag{9}$$

and

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty. \tag{10}$$

then the series $\Sigma A_n(t) P_n \lambda_n (np_n)^{-1}$ is summable $|\bar{N}, p_n|_k$ for $k \geqslant 1$.

Theorem B. *Let the sequence (p_n) be such that conditions (7) and (8) of Theorem A are satisfied. If Σa_n is a series of complex terms such that*

$$B_n \equiv \sum_{v=1}^{n} v a_v = O(n), \tag{11}$$

then the series $\Sigma a_n P_n \lambda_n (np_n)^{-1}$ is summable $|\bar{N}, p_n|_k$ for $k \geqslant 1$.

3. The aim of this paper is to prove above theorems for $|\bar{N}, p_n; \delta|_k$, with $k \geqslant 1$ and $\delta \geqslant 0$, summability. Now, we shall prove the following theorem.

Theorem 1. *Let the sequence (p_n) be such that conditions (7) and (8) of Theorem A are satisfied and*

$$\sum_{n=v}^{\infty} (P_n/p_n)^{\delta k - 1} \frac{1}{P_{n-1}} = O\left\{(P_v/p_v)^{\delta k} \frac{1}{P_v}\right\}. \tag{12}$$

If $\varphi_1(t)$ is of bounded variation in $(0,\pi)$ and (λ_n) is a sequence such that

$$\sum_{n=1}^{\infty} n^{\delta k - 1} |\lambda_n|^k < \infty \tag{13}$$

and

$$\sum_{n=1}^{\infty} n^{\delta k} |\Delta\lambda_n| < \infty, \tag{14}$$

then the series $\Sigma A_n(t) P_n \lambda_n (np_n)^{-1}$ is summable $|\bar{N}, p_n; \delta|_k$ for $k \geqslant 1$ and $\delta \geqslant 0$.

Theorem 2. *Let the sequence (p_n) be such that conditions* (7) *and* (8) *of Theorem A and condition* (12) *of Theorem* 1 *are satisfied. If condition* (11) *of Theorem B is satisfied by the series Σa_n, then the series $\Sigma a_n P_n \lambda_n (np_n)^{-1}$ is summable $|\bar{N},p_n;\delta|_k$ for $k \geqslant 1$ and $\delta \geqslant 0$, where (λ_n) is as in Theorem* 1.

4. We need the following lemmas for the proof of our theorems.

Lemma 1. *If $\varphi_1(t)$ is of bounded variation in $(0,\pi)$, then*

$$\sum_{v=1}^{n} vA_v(x) = O(n) \text{ as } n \to \infty. \tag{15}$$

This lemma is a particular case of Lemma due to Prasad and Bhatt ([5], Lemma 9).

Lemma 2. ([3]). *If the sequence (p_n) is such that conditions* (7) *and* (8) *of Theorem A are satisfied, then*

$$\Delta\{P_n/(p_n n^2)\} = O(1/n^2) \text{ as } n \to \infty. \tag{16}$$

5. *Proof of Theorem* 2. Let (T_n) denote the $(\bar{N},p_n)$ mean of the series $\Sigma a_n P_n \lambda_n (np_n)^{-1}$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i P_i \lambda_i (ip_i)^{-1} = \frac{1}{P_n}\sum_{v=0}^{n} (P_n - P_{v-1}) a_v P_v \lambda_v (vp_v)^{-1}.$$

Then, for $n \geqslant 1$, we have that

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} P_v a_v \lambda_v (vp_v)^{-1}.$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= B_n \lambda_n n^{-2} - p_n(P_n P_{n-1})^{-1} \sum_{v=1}^{n-1} p_v P_v B_v \lambda_v (v^2 p_v)^{-1} \\ &\quad + p_n(P_n P_{n-1})^{-1} \sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v B_v (v^2 p_v)^{-1} \\ &\quad + p_n(P_n P_{n-1})^{-1} \sum_{v=1}^{n-1} P_v B_v \lambda_{v+1} \Delta\{P_v/(v^2 p_v)\} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.} \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality for $k > 1$, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1} |T_{n,i}|^k < \infty, \text{ for } i = 1,2,3,4.$$

Firstly, we have that

$$\begin{aligned}
\sum_{n=1}^{m} (P_n/p_n)^{\delta k+k-1} |T_{n,1}|^k &= \sum_{n=1}^{m} (P_n/p_n)^{\delta k} (P_n/p_n)^{k-1} |\lambda_n|^k |B_n|^k n^{-2k} \\
&= O(1) \sum_{n=1}^{m} n^{\delta k} n^{k-1} |\lambda_n|^k n^k n^{-2k} \\
&= O(1) \sum_{n=1}^{m} n^{\delta k-1} |\lambda_n|^k \\
&= O(1) \text{ as } m \to \infty,
\end{aligned}$$

by virtue of (7), (11), (13).

Now, when $k > 1$ applying Hölder's inequality, with indices k and k', where $1/k + 1/k' = 1$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} |T_{n,2}|^k &= \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} \left| \sum_{v=1}^{n-1} \frac{p_v P_v B_v \lambda_v}{v^2 p_v} \right|^k \\
&\leqslant \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \left\{ \frac{P_v |B_v| |\lambda_v|}{v^2 p_v} \right\}^k \\
&\quad \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^k (P_v/p_v)^k |B_v|^k v^{-2k} \\
&\quad \times \sum_{n=v+1}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} (P_v/p_v)^{\delta k} (P_v/p_v)^{k-1} |B_v|^k |\lambda_v|^k v^{-2k} \\
&= O(1) \sum_{v=1}^{m} v^{\delta k-1} |\lambda_v|^k \\
&= O(1)
\end{aligned}$$

as $m \to \infty$, by (7), (11), (12) and (13).

On the other hand, since

$$\sum_{v=1}^{n-1} P_v |\Delta\lambda_v| \leqslant P_{n-1} \sum_{v=1}^{n-1} |\Delta\lambda_v| \Rightarrow \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta\lambda_v| \leqslant \sum_{v=1}^{n-1} |\Delta\lambda_v| = O(1),$$

by (14), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} |T_{n,3}|^k &\leqslant \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} \left| \sum_{v=1}^{n-1} \frac{P_v P_v B_v \Delta\lambda_v}{v^2 p_v} \right|^k \\
&\leqslant \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} P_v |\Delta\lambda_v| \right.
\end{aligned}$$

$$\times (P_v/p_v)^k |B_v|^k v^{-2k} \Big\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta\lambda_v| \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} P_v |\Delta\lambda_v| (P_v/p_v)^k |B_v|^k v^{-2k}$$

$$\times \sum_{n=v+1}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{v-1}}$$

$$= O(1) \sum_{v=1}^{m} (P_v/p_v)^{\delta k} (P_v/p_v)^k |B_v|^k v^{-2k} |\Delta\lambda_v|$$

$$= O(1) \sum_{v=1}^{m} v^{\delta k} |\Delta\lambda_v|$$

$$= O(1),$$

as $m \to \infty$, by virtue of (7), (11), (12) and (14).

Finally, using the fact that $\Delta\{P_v/(v^2 p_v)\} = O(1/v^2)$, by Lemma 2, we have

$$\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} |T_{n,4}|^k \leqslant \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k}$$

$$\times \sum_{v=1}^{n-1} P_v |B_v| |\lambda_{v+1}| \Delta\{(P_v/(v^2 p_v)\}|^k$$

$$= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} \Big\{ \sum_{v=1}^{n-1} p_v (P_v/p_v) v^{-2}$$

$$\times |B_v| |\lambda_{v+1}| \Big\}^k$$

$$= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \Big\{ \sum_{v=1}^{n-1} (P_v/p_v)^k p_v v^{-2k}$$

$$\times |B_v|^k |\lambda_{v+1}|^k \Big\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} (P_v/p_v)^k p_v |\lambda_{v+1}|^k |B_v|^k v^{-2k}$$

$$\times \sum_{n=v+1}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} (P_v/p_v)^{\delta k} (P_v/p_v)^{k-1} |\lambda_{v+1}|^k v^k v^{-2k}$$

$$= O(1) \sum_{v=1}^{m} v^{\delta k-1} |\lambda_{v+1}|^k$$

$$= O(1)$$

as $m \to \infty$, by (7), (11), (12) and (13). Therefore, we get that

$$\sum_{n=1}^{m} (P_n/p_n)^{\delta k+k-1} |T_{n,i}|^k = O(1) \text{ as } m \to \infty, \text{ for } i = 1, 2, 3, 4.$$

This completes the proof of Theorem 2.

Proof of Theorem 1. Theorem 1 is a direct consequence of Theorem 2 and Lemma 1.

Remark. If we take $\delta = 0$ in our theorems 1 and 2, then we get Theorem A and Theorem B, respectively. Because in this case the conditions (13) and (14) reduce to conditions (9) and (10), respectively. It should be noted that in this case condition (12) is obvious.

If we take $p_n = 1$ for all values of n in Theorem 1, then we get the following corollary.

COROLLARY

If $\varphi_1(t)$ is of bounded variation in $(0, \pi)$ and (λ_n) is a sequence such that conditions (13) *and* (14) *of Theorem* 1 *are satisfied, then the series $\Sigma A_n(t)\lambda_n$, at $t = x$ is summable $|C, 1; \delta|_k, k \geqslant 1$, provided that $1 - \delta k > 0$.*

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Maximal monotone differential inclusions with memory

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Abstract. In this paper we study maximal monotone differential inclusions with memory. First we establish two existence theorems; one involving convex-valued orientor fields and the other nonconvex valued ones. Then we examine the dependence of the solution set on the data that determine it. Finally we prove a relaxation theorem.

Keywords. Maximal monotone operator; resolvent; resolvent convergence topology; selection theorem; relaxation.

1. Introduction

In this paper we examine maximal monotone differential inclusions with memory, defined in $\mathbb{R}^N$. First we consider the existence problem, and we prove two such theorems. One with a convex-valued orientor field and the other with a nonconvex valued one. Then we examine the dependence of the solution set on the data that determine it; i.e., the maximal monotone operator, the orientor field and the past history information. More precisely, we consider a parametrized family of problems, where all the above data depend on the parameter, and we examine how the solution set responds to variations of the parameter. Finally we prove a "relaxation" result, which says that under reasonable hypotheses on the orientor field, the solution set of the "nonconvex problem" is dense in that of the "convex problem". Our formulation of the problem is general enough to incorporate subdifferential systems. Among them of particular interest, because of their diverse applications, are those for which the maximal monotone operator $A = \partial\delta_K$, with δ_K being the indicator function of a nonempty, closed and convex subset K of $\mathbb{R}^N$(i.e., $\delta_K(x) = 0$ if $x \in K$ and $\delta_K(x) = +\infty$ if $x \notin K$) and $\partial\delta_K(\cdot)$ denotes its subdifferential in the sense of convex analysis. It is well-known (see for example Aubin-Cellina [2]), that $\partial\delta_K(x) = N_K(x)$ for every $x \in K$, with $N_K(x)$ being the normal cone to the set K at x. In this case, the corresponding "differential inclusion" is also called "differential variational inequality" and appears in mathematical economics, in the study of dynamic allocation processes (see Aubin-Cellina [2], Henry [10] and Stacchetti [18]) and in theoretical mechanics in the study of unilateral processes (see Moreau [14]). Our system has a memory feature, since the derivative of the state depends on the past history of it. We should mention, that this memory feature of our system, arises in the so-called "absorption lag" dynamic economic models. It signifies that the growth rate $\dot{x}(t)$ of the capital depends on the past history $x_t(\cdot)$ of the capital. Finally given that every control system, after

"deparametrization" (union over all admissible controls of all vector fields), can be described by a differential inclusion, the systems studied in this paper incorporate hereditary control systems, monitored by maximal monotone, multivalued in general operators. Our results also extend the works on differential inclusions done by Aubin-Cellina [2], Bressan [5] and Cellina-Marchi [7].

2. Preliminaries

Let (Ω, Σ) be a measurable space and let X be a separable Banach space. Throughout this paper, we will be using the following notation: $P_{f(c)}(X)=\{A\subseteq X$: nonempty, closed (convex)$\}$. A multifunction (set-valued function) $F:\Omega\to P_f(X)$, is said to be measurable, if for all $x\in X$, the $\mathbb{R}_+$-valued function $\omega\to d(x,F(\omega))=\inf\{\|x-z\|:z\in F(\omega)\}$ is measurable. Other equivalent definitions of measurability of a $P_f(X)$-valued multifunction can be found in Wagner [20] (see theorem 4.2). We will say that $G:\Omega\to 2^X\setminus\{\emptyset\}$ is graph-measurable, provided that $\mathrm{Gr}G=\{(\omega,x)\in\Omega\times X:x\in F(\omega)\}\in\Sigma x B(X)$, with $B(X)$ being the Borel σ-field of X. For $P_f(X)$-valued multifunctions, measurability implies graph measurability, while the converse is true if there exists a σ-finite measure $\mu(\cdot)$ on Σ, with respect to which Σ is complete (see Wagner [20], theorem 4.2). Now let (Ω,Σ,μ) be a finite measure space and $F:\Omega\to 2^X\setminus\{\emptyset\}$. By S^1_F we will denote the set of integrable selectors of $F(\cdot)$; i.e., $S^1_F=\{f\in L^1(X):f(\omega)\in F(\omega)\mu-\text{a.e.}\}$. This set may be empty. For a graph measurable multifunction $F(\cdot)$, it is nonempty if and only if $\omega\to\inf\{\|z\|:z\in F(\omega)\}\in L^1_+$. In particular, this is the case if $\omega\to|F(\omega)|=\sup\{\|z\|:z\in F(\omega)\}\in L^1_+$. Such multifunctions are called "integrably bounded".

Next let Y,Z be Hausdorff topological spaces and $G: Y\to 2^Z\setminus\{\emptyset\}$ a multifunction. We will say that $G(\cdot)$ is upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c.)), if for every $U\subseteq Z$ open $G^+(U)=\{y\in Y:G(y)\subseteq U\}$ (resp. $G^-(U)=\{y\in Y:G(y)\cap U\neq\emptyset\}$) is open in Y. A multifunction $G(\cdot)$ which is both u.s.c. and l.s.c. is said to be continuous. So a continuous multifunction $G(\cdot)$, is one that is continuous from Y into $2^Z\setminus\{\emptyset\}$ equipped with the Vietoris topology (see Klein-Thompson [12]). If Z is a metric space, then on $P_f(Z)$ we can define a (generalized) metric, known in the literature as the Hausdorff metric, by setting $h(A,B)=\max[\sup_{a\in A}d(a,B),\sup_{b\in B}d(b,A)]$, for every $A,B\in P_f(Z)$. If Z is complete, then so is the metric space $(P_f(Z),h)$. A multifunction $F:\Omega\to P_f(Z)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from Y into the metric space $(P_f(Z),h)$. If the multifunction has nonempty compact values, then continuity and h-continuity coincide. This follows from the fact that on the collection of nonempty compact sets of a metric space, the Vietoris and Hausdorff topologies coincide (see Klein-Thompson [12], corollary 4.2.3, p. 41).

Let V be a Banach space and $\{A_n,A\}_{n\geqslant 1}\subseteq 2^V\setminus\{\emptyset\}$. Denote by s- the strong topology on V and by w- the weak topology. We define:

$$s\text{-}\underline{\lim}\,A_n=\{x\in V:x=s\text{-}\lim x_n, x_n\in A_n, n\geqslant 1\}=\{x\in V:\lim d(x,A_n)=0\}$$

$$\begin{aligned}s\text{-}\overline{\lim}\,A_n&=\{x\in V:x=s\text{-}\lim x_{n_k}, x_{n_k}\in A_{n_k}, n_1<n_2<\cdots<n_k<\cdots\}\\&=\{x\in V:\underline{\lim}\,d(x,A_n)=0\}\end{aligned}$$

and

$$w\text{-}\overline{\lim}\,A_n=\{x\in V:x=w\text{-}\lim x_{n_k}, x_{n_k}\in A_{n_k}, n_1<n_2<\cdots<n_k<\cdots\}.$$

It is clear from the above definitions, that we always have $s\text{-}\underline{\lim}\, A_n \subseteq s\text{-}\overline{\lim}\, A_n \subseteq w\text{-}\overline{\lim}\, A_n$. If $s\text{-}\underline{\lim}\, A_n = w\text{-}\overline{\lim}\, A_n = A$, then we say that the A_n's converge to A in the Kuratowski-Mosco sense, denoted by $A_n \xrightarrow{K-M} A$ (see Mosco [15]). If $\dim V < \infty$, then the weak and strong topologies coincide, and so we recover the well-known Kuratowski mode of set convergence (see Kuratowski [13]). If $s\text{-}\underline{\lim}\, A_n = A = s\text{-}\overline{\lim}\, A_n$, then we write $A_n \xrightarrow{K} A$.

Let us also recall some basic facts about maximal monotone operators. So let H be a Hilbert space. An operator $A: D(A) \subseteq H \to 2^H$ is said to be "monotone", if and only if $(x - x', y - y') \geqslant 0$ for all $[x, y], [x', y'] \in \mathrm{Gr}\, A$ (here $(\cdot,\cdot)$ denotes the inner product in H). It is said to be maximal monotone if and only if $(x - v, y - w) \geqslant 0$ for all $[x, y] \in \mathrm{Gr}\, A$, which implies that $w \in Av$ (i.e., the graph of A is not properly included in any other monotone subset of $H \times H$). From a well-known theorem of Minty, we have that $A(\cdot)$ is maximal monotone if and only if from some $\lambda > 0$, $R(I + \lambda A) = H$. Then for every $\lambda > 0$, $J_\lambda = (I + \lambda A)^{-1}: R(I + \lambda A) = H \to D(A)$, and is called the "resolvent of A". The resolvent $J_\lambda(\cdot)$ is nonexpansive and $J_\lambda x \xrightarrow{s} x$ as $\lambda \to 0^+$ for each $x \in \overline{D(A)}$. Let $\mathscr{M}$ be the set of all maximal monotone operators in H. The topology of R-convergence on $\mathscr{M}$, is the weakest topology, that makes continuous the maps $\hat{J}_{\lambda,x}: \mathscr{M} \to H$ for every $\lambda > 0$ and $x \in H$, where $\hat{J}_{\lambda,x}(A) = (I + \lambda A)^{-1} x$. We will denote by $\mathscr{M}_R$ (or $\mathscr{M}_R(H)$), the set $\mathscr{M}$ equipped with the topology of R-convergence. If H is separable, then $\mathscr{M}_R$ is a Polish space (i.e., a separable, metrizable, complete space). Furthermore, we know that $A_n \xrightarrow{\mathscr{M}_R} A$ if and only if $\mathrm{Gr}\, A_n \xrightarrow{K-M} \mathrm{Gr}\, A$. For further details, we refer to Attouch [1]. Finally, note that if $A(\cdot)$ is maximal monotone, for every $x \in D(A)$, Ax is closed and convex. Hence for every $x \in D(A)$, Ax contains an element of minimum norm (the projection of the origin on Ax). This unique element is denoted by $A^0 x$. Thus we have $A^0 x \in Ax$ and $\|A^0 x\| = \inf\{\|y\| : y \in Ax\}$. The single-valued operator $A^0: D(A) \to H$, is called the "minimal section" of A.

3. Existence theorem

Let $b, r > 0$ and set $\hat{T} = [-r, b]$, $T_0 = [-r, 0]$ and $T = [0, b]$. We will be studying the following maximal monotone differential inclusion with memory:

$$\left.\begin{aligned} -\dot{x}(t) &\in Ax(t) + F(t, x_t) \text{ a.e. on } T \\ x(v) &= \varphi(v) \quad v \in T_0. \end{aligned}\right\} \qquad (*)$$

Here $x_t \in C(T_0, \mathbb{R}^N)$ and is defined by $x_t(s) = x(t+s)$. So $x_t(\cdot)$ gives us the history of state $x(\cdot)$ from $t - r$ up to the present time t.

In this section we present two existence results concerning $(*)$. The first assumes that the multivalued perturbation $F(t, x_t)$ is convex valued, while the second that it is nonconvex valued.

For the first existence theorem, we will need the following hypotheses on the data.

$\underline{H(A)}$: $A: D(A) \subseteq \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone operator.

$\underline{H(F)}$: $F: T \times C(T_0, \mathbb{R}^N) \to P_{fc}(\mathbb{R}^N)$ is a multifunction s.t.

(1) $t \to F(t, y)$ is measurable,

(2) $y \to F(t, y)$ is u.s.c.,

(3) $|F(t, y)| = \sup\{\|v\| : v \in F(t, y)\} \leqslant a(t) + b(t)\|y\|_\infty$ a.e. with $a(\cdot), b(\cdot) \in L^1_+$.

$\underline{H(\varphi)}$: $\varphi \in C(T_0, \mathbb{R}^N)$, $\varphi(0) \in \overline{D(A)}$.

By a solution of $(*)$, we understand a function $x \in C(\hat{T}, \mathbb{R}^N)$ s.t. $x(v) = \varphi(v)$ for $v \in T_0$ and $x|_T$ is a solution of the initial value problem $-\dot{x}(t) \in Ax(t) + F(t, x_t)$ a.e., $x(0) = \varphi(0)$ (i.e., $x: T \to \mathbb{R}^N$ is absolutely continuous and there exists $f \in S^1_{F(\cdot, x.)}$ *s.t.* $-\dot{x}(t) \in Ax(t) + f(t)$ a.e., $x(0) = \varphi(0)$).

Theorem 3.1. *If hypotheses $H(A), H(F)$ and $H(\varphi)$ hold, then $(*)$ admits a solution and the solution set is compact in $C(\hat{T}, \mathbb{R}^N)$.*

Proof. First we will obtain an a priori, uniform bound for the solutions of $(*)$. So let $x(\cdot) \in C(\hat{T}, \mathbb{R}^N)$ be such a solution. Then by definition we can find $f(\cdot) \in L^1(T, \mathbb{R}^N)$, $f(t) \in F(t, x_t)$ a.e. s.t. $-\dot{x}(t) \in Ax(t) + f(t)$ a.e. From Benilan's inequality (see for example Vrabie [19], corollary 1.7.1, p. 35), we have that

$$\| x(t) \| \leqslant \| S(t)\varphi(0) \| + \int_0^t \| f(s) \mathrm{d}s,$$

where $\{S(t)\}_{t \in T}$ is the nonlinear semigroup of contractions generated by the maximal operator $A(\cdot)$. Recalling that $t \to \| S(t)\varphi(0) \|$ is continuous on T, we can find $M > 0$ s.t. $\| S(t)\varphi(0) \| \leqslant M$ for all $t \in T$. Hence using growth hypothesis $H(F)(3)$, we get

$$\| x(t) \| \leqslant M + \int_0^t (a(s) + b(s) \| x_s \|_\infty) \mathrm{d}s.$$

Let $h(t) = \| x_t \|_\infty$. Then clearly $h(\cdot) \in C(T, \mathbb{R}^N)$ and we have

$$h(t) \leqslant \hat{M} + \int_0^t (a(s) + b(s)h(s)) \mathrm{d}s, \quad t \in T,$$

with $\hat{M} = \max[M, \| \varphi \|_\infty]$. Invoking Gronwall's inequality we get $M_1 > 0$ s.t. for all $t \in T$ we have

$$h(t) = \| x_t \|_\infty \leqslant M_1.$$

Then consider the following, modified orientor field

$$\hat{F}(t, y) = \begin{matrix} F(t, y) \text{ if } \| y \|_\infty \leqslant M_1 \\ F\left(t, \dfrac{M\hat{y}}{\| y \|_\infty}\right) \text{ if } \| y \|_\infty > M_1. \end{matrix}$$

Note that $\hat{F}(t, y) = F(t, p_{M_1}(y))$, where $p_{M_1}(\cdot)$ is the M_1-radial retraction on the Banach space $C(T_0, \mathbb{R}^N)$. Hence, $t \to \hat{F}(t, y)$ is measurable, while since $p_{M_1}(\cdot)$ is Lipschitz continuous, $y \to \hat{F}(t, y)$ is u.s.c. (see Klein-Thompson [12], theorem 7.3.11, p. 87). Furthermore we have:

$$|\hat{F}(t, y)| = \sup\{ \| v \| : v \in \hat{F}(t, y) \} \leqslant a(t) + b(t)M_1 = \psi(t) \text{ a.e.}$$

with $\psi(\cdot) \in L^1_+$. Then we consider problem $(*)$ with $\hat{F}(t, y)$ instead of $F(t, y)$. Set $V = \{ g \in L^1(T, \mathbb{R}^N) : \| g(t) \| \leqslant \psi(t) \text{ a.e.} \}$. From the Dunford-Pettis theorem, we know that V is sequentially weakly compact. In what follows V will be equipped with the relative weak-$L^1(T, \mathbb{R}^N)$ topology. Let $R: V \to P_{fc}(V)$ be the multifunction defined by

$R(g) = S^1_{\hat{F}(\cdot, p(g).)}$, where $p: L^1(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$ is the map that to each $g \in L^1(T, \mathbb{R}^N)$ assigns the unique solution of $-\dot{x}(t) \in Ax(t) + g(t)$a.e., $x(0) = \varphi(0)$ (its existence is guaranteed by the Benilan-Brezis theorem; see Vrabie [19], theorem 1.9.1, p. 41). We claim that $R(\cdot)$ is an u.s.c. multifunction. Given that V equipped with the relative weak-$L^1(T, \mathbb{R}^N)$ topology is compact, metrizable (see Dunford-Schwartz [8], theorem 3, p. 434), it is enough to show that $\operatorname{Gr} R$ is sequentially closed in $V \times V$ (see Klein-Thompson [12], theorem 7.1.16, p. 78). To this end, let $[g_n, f_n] \in \operatorname{Gr} R n \geqslant 1$ and assume that $[g_n, f_n] \to [g, f]$ in $V \times V$. From corollary 2.3.1, p. 67 of Vrabie [19], we know that $p(\cdot)$ is sequentially continuous from $L^1(T, \mathbb{R}^N)$ equipped with the weak topology into $C(T, \mathbb{R}^N)$ with the strong topology. So if we define $\hat{p}(g)(\cdot) \in C(\hat{T}, \mathbb{R}^N)$ by setting $\hat{p}(g)(t) = p(g)(t)$ for $t \in T$ and $\hat{p}(g)(v) = \varphi(v)$ for $v \in T_0$, then we have $\hat{p}(g_n)_t \xrightarrow{s} \hat{p}(g)_t$ in $C(T_0, \mathbb{R}^N)$ for all $t \in T$. Hence applying theorem 4.2. of [16] we have

$$
\begin{aligned}
& f \in w\text{-}\overline{\lim}\, S^1_{\hat{F}(\cdot, \hat{p}(g).)} \subseteq S^1_{\hat{F}(\cdot, \hat{p}(g)\cdot)} \\
& \Rightarrow [g, f] \in \operatorname{Gr} R \\
& \Rightarrow R(\cdot) \text{ is indeed u.s.c. as claimed.}
\end{aligned}
$$

Since $R(\cdot)$ is closed, convex valued we can apply the Kakutani-KyFan fixed point theorem to get $g \in R(g)$. Let $\hat{p}(g)(\cdot) = x(\cdot) \in C(\hat{T}, \mathbb{R}^N)$. Then this function solves $(*)$ with $\hat{F}(t, y)$ instead of $F(t, y)$. But as in the beginning of the proof, via Gronwall's inequality (see Vrabie [19], p. 3) we can get that $\|x_t\|_\infty \leqslant M_1$. Hence $\hat{F}(t, x_t) = F(t, x_t)$. Thus $x(\cdot)$ solves $(*)$.

Since the solution set of $(*)$ lies in $\hat{p}(V)$ and the latter is compact in $C(\hat{T}, \mathbb{R}^N)$, to establish the compactness of the solution set, it suffices to show that it is closed. So let $\{x_n\}_{n \geqslant 1} \subseteq C(\hat{T}, \mathbb{R}^N)$ be solutions of $(*)$ and assume that $x_n \to x$ in $C(\hat{T}, \mathbb{R}^N)$. Then $x_n = \hat{p}(f_n)$ for some $f_n \in S^1_{F(\cdot, (x_n).)}$. Because of hypothesis $H(F)(3)$ and the Dunford-Pettis theorem, we have that $\{f_n\}_{n \geqslant 1}$ is relatively sequentially w-compact in $L^1(T, \mathbb{R}^N)$. So by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(T, \mathbb{R}^N)$ and so $\hat{p}(f_n) \to \hat{p}(f) = x$ in $C(\hat{T}, \mathbb{R}^N)$. Also from theorem 4.2 of [16], we have that $f \in S^1_{\hat{F}(\cdot, x.)}$. Hence $x(\cdot) = \hat{p}(f)(\cdot)$ is also a solution of $(*)$, establishing the compactness in $C(\hat{T}, \mathbb{R}^N)$ of $(*)$. **Q.E.D.**

We can also have an existence result for the case where the orientor field $F(\cdot, \cdot)$ is not necessarily convex-valued. For this we will need the following hypotheses on the $F(t, y)$.

<u>$H(F)_1$:</u> $F: T \times C(T_0, \mathbb{R}^N) \to P_f(\mathbb{R}^N)$ is a multifunction s.t.

- **(1)** $(t, y) \to F(t, y)$ is graph measurable,
- **(2)** $y \to F(t, y)$ is l.s.c.,
- **(3)** $|F(t, y)| \leqslant a(t) + b(t)\|y\|_\infty$ a.e., with $a(\cdot), b(\cdot) \in L^1_+$.

Theorem 3.2. *If hypotheses $H(A)$, $H(F)_1$, and $H(\varphi)$ hold, then $(*)$ admits a solution.*

Proof. Let $V \subseteq L^1(T, \mathbb{R}^N)$ and $\hat{F}(t, y)$ be defined as in the proof of theorem 3.1. Let $K = \hat{p}(V)$ (i.e., $K = p(V)$ on T and $K = \{\varphi\}$ on T_0). From the continuity property of $\hat{p}(\cdot)$ (see the proof of theorem 3.1), we have that K is compact in $C(\hat{T}, \mathbb{R}^N)$. Hence by Mazur's theorem so is $\hat{K} = \overline{\operatorname{conv}} K$. Let $R: \hat{K} \to P_f(L^1(T, \mathbb{R}^N))$ be defined by $R(y) = S^1_{\hat{F}(\cdot, y.)}$. Using theorem 4.1 of [16], we get that $R(\cdot)$ is l.s.c. Applying Fryszkowski's

selection theorem [9], we get $r:\hat{K}\to L^1(T,\mathbb{R}^N)$ continuous *s.t.* $r(y)\in R(y)$ for all $y\in\hat{K}$. Then let $q:\hat{K}\to\hat{K}$ be defined by $q=\hat{p}\circ r$. Clearly $q(\cdot)$ is continuous. So applying Schauder's fixed point theorem, we get $x\in\hat{K}$ s.t. $x=q(x)$. Again through the definition of $\hat{F}(t,y)$ and Gronwall's inequality, we can check that $\|x_t\|_\infty \leqslant M_1 \Rightarrow \hat{F}(t,x_t)=F(t,x_t)\Rightarrow x(\cdot)\in C(\hat{T},\mathbb{R}^N)$ solves $(*)$. **Q.E.D.**

Remark. Theorem 3.2 above, improves the result of Cellina-Marchi [7], who considered memoryless systems and assumed that the orientor field was h-continuous in both variables, also extends the existence result of Bressan [5], who also considered memoryless systems and assumed that $A\equiv 0$.

4. A continuous dependence result

In this section, we investigate the dependence of the solution set of $(*)$ on the data that determine it; namely the maximal monotone operator, the orientor field and the function φ.

So let E be a metric space. We consider the following family of problems parametrized by elements in E:

$$\left.\begin{array}{l} -\dot{x}(t)\in A(r)x(t)+F(t,x_t,r) \text{ a.e. on } T \\ x(v)=\varphi(r)(v)\ v\in T_0,\ r\in E. \end{array}\right\} (*)_r$$

We denote the solution set of $(*)_r$ by $P(r)\subseteq C(\hat{T},\mathbb{R}^N)$. We want to examine the dependence of $P(\cdot)$ on $r\in E$.

To this end we will need the following two auxiliary results. Recall that if $A:D(A)\subseteq\mathbb{R}^N\to 2^{\mathbb{R}^N}$ is a maximal monotone operator, then we can define the realization of A on $L^2(T,\mathbb{R}^N)$, $\hat{A}:D(\hat{A})\subseteq L^2(T,\mathbb{R}^N)\to 2^{L^2(T,\mathbb{R}^N)}$ by

$$\hat{A}x=\{y\in L^2(T,\mathbb{R}^N):y(t)\in Ax(t) \text{ a.e. on } T\}$$

for each $x\in D(A)=\{v\in L^2(T,\mathbb{R}^N):v(t)\in D(A)$ a.e. and there exists $\omega\in L^2(T,\mathbb{R}^N)$ s.t. $\omega(t)\in Av(t)$ a.e.$\}$. It is well-known and easy to prove that the realization $\hat{A}(\cdot)$ is maximal monotone too.

Lemma 4.1. *If $A:E\to\mathcal{M}_R(\mathbb{R}^N)$ is continuous, then so is $\hat{A}:E\to\mathcal{M}_R(L^2(T,\mathbb{R}^N))$.*

Proof. Let $r_n\to r$ in E and let $s(t)=\Sigma_{k=1}^m \chi_{C_k}\omega_k$, with $C_k\in\Sigma$, $\omega_k\in\mathbb{R}^N$ (a simple function). Then since by hypothesis $A(r_n)\to A(r)$ in $\mathcal{M}_r(\mathbb{R}^N)$, we have

$$J_\lambda^{A(r_n)}\omega_k\to J_\lambda^{A(r)}\omega_k \text{ as } n\to\infty$$

for all $k\in\{1,\dots,m\}$, $\lambda>0$. Thus

$$\sum_{k=1}^m \chi_{C_k}(t)J_\lambda^{A(r_n)}\omega_k\to\sum_{k=1}^m \chi_{C_k}(t)J_\lambda^{A(r)}\omega_k \text{ as } n\to\infty$$

for all $t\in T$. From this we deduce that

$$J_\lambda^{\hat{A}(r_n)}(s)\xrightarrow{s}J_\lambda^{\hat{A}(r)}(s) \text{ as } n\to\infty \text{ in } L^2(T,\mathbb{R}^N)$$

for all $\lambda > 0$. Since $s(\cdot)$ was an arbitrary simple function, simple functions are dense in $L^2(T, \mathbb{R}^N)$ and the resolvent operator is nonexpansive, we get

$$J_\lambda^{\hat{A}(r_n)}(x) \xrightarrow{s} J_\lambda^{\hat{A}(r)}(x) \text{ as } n \to \infty \text{ in } L^2(T, \mathbb{R}^N)$$

for all $x \in L^2(T, \mathbb{R}^N)$ and all $\lambda > 0 \Rightarrow \hat{A}(r_n) \to \hat{A}(r)$ in $\mathcal{M}_R(L^2(T, \mathbb{R}^N))$ (see § 2) $\Rightarrow \hat{A}: E \to \mathcal{M}_R(L^2(T, \mathbb{R}^N))$ is indeed continuous. **Q.E.D.**

The second auxiliary result that we will need is the following:

Lemma 4.2. If X is a Banach space, $\{A_n\}_{n \geq 1} \subseteq P_f(X)$, $A_n \subseteq K$ for all $n \geq 1$ with $K \subseteq X$ compact, and $A_n \xrightarrow{K} A$ as $n \to \infty$, then $A_n \xrightarrow{h} A$ as $n \to \infty$.

Proof. Let $a_n \in A_n$, $n \geq 1$ s.t. $d(a_n, A) = \sup_{b \in A_n} d(b, A)$. It exists since $A_n, n \geq 1$ is compact (being a closed subset of the compact set K). Note that $\{a_n\}_{n \geq 1} \subseteq K$. So by passing to a subsequence if necessary, we may assume that $a_n \to a$. Because $A_n \xrightarrow{K} A$, we have $a \in A$. Then $d(a_n, A) \to d(a, A) = 0 \Rightarrow \sup_{b \in A_n} d(b, A) \to 0$ as $n \to \infty$.

Similarly, let $a_n \in A$ $n \geq 1$ s.t. $\sup_{b \in A} d(b, A_n) = d(a_n, A_n)$. Since A is compact and $\{a_n\}_{n \geq 1} \subseteq A$, we may assume that $a_n \to a \in A$. Then we have

$$|d(a_n, A_n) - d(a, A_n)| \leq \|a_n - a\| \to 0.$$

But since $A_n \xrightarrow{K} A$, we have $d(a, A_n) \to 0$. So $d(a_n, A_n) \to 0$ as $n \to \infty \Rightarrow \sup_{b \in A} d(b, A_n) \to 0 \Rightarrow A_n \xrightarrow{h} A$. **Q.E.D.**

To prove our continuous dependence result, we will need the following hypotheses on the data.

<u>$H(A)_1$:</u> $A: E \to \mathcal{M}_R(\mathbb{R}^N)$ is continuous, for every $r \in E$ $D(A(r))$ is closed and $A^0(r)$ is bounded on compact subsets of $D(A(r))$, uniformly in $r \in B \subseteq E$ nonempty, compact.

Remark. This hypothesis is clearly satisfied if $A(r) = \partial \delta_{K(r)}$ with $K: E \to P_{fc}(\mathbb{R}^N)$ continuous, or if for every $r \in E$, $D(A(r)) = \mathbb{R}^N$ and $r \to A^0(r)$ is bounded on compact sets.

<u>$H(F)_2$:</u> $F: T \times C(T_0, \mathbb{R}^N) \times E \to P_{fc}(\mathbb{R}^N)$ is a multifunctions s.t.
(1) $t \to F(t, y, r)$ is measurable,
(2) $(y, r) \to F(t, y, r)$ is continuous,
(3) $h(F(t, y, r), F(t, y', r)) \leq \eta(t) \|y - y'\|_\infty$ a.e. with $\eta(\cdot) \in L_+^1$,
(4) $|F(t, y, r)| \leq a_B(t) + b_B(t) \|y\|_\infty$ with $a_B(\cdot), b_B(\cdot) \in L_+^2$ and for all $r \in B \subseteq E$ nonempty, compact.

<u>$H(\varphi)_1$:</u> $\varphi: E \to C(T_0, \mathbb{R}^N)$ is continuous and for all $r \in E$ $\varphi(r)(0) \in \overline{D(A(r))}$.

Theorem 4.3. *If hypotheses $H(A)_1, H(F)_2$ and $H(\varphi)_1$ hold, then $r \to P(r)$ from E into the nonempty compact subsets of $C(\hat{T}, \mathbb{R}^N)$ is continuous and h-continuous.*

Remark. The hypotheses of this theorem and theorem 3.1 guarantee that for every $r \in E$, $P(r)$ is nonempty and compact in $C(\hat{T}, \mathbb{R}^N)$.

Proof. Let $r_n \to r$ in E and let $x \in s\text{-}\overline{\lim}\, P(r_n)$. Then by denoting subsequences with the same index as the original sequences for economy in the notation, we know that there exist $x_n \in P(r_n)$ $n \geqslant 1$ s.t. $x_n \to x$ in $C(\hat{T}, \mathbb{R}^N)$. Then by definition

$$
\begin{aligned}
&-\dot{x}_n(t) \in A(r_n)x_n(t) + f_n(t) \text{ a.e. on } T\\
&x_n(v) = \varphi(r_n)(v) \quad v \in T_0,
\end{aligned}
$$

where $f_n \in L^1(T, \mathbb{R}^N)$ and $f_n(t) \in F(t, (x_n)_t, r_n)$ a.e.. Then from the Benilan-Brezis theorem (see Vrabie [19], theorem 1.9.1, p. 41), we have

$$
\begin{aligned}
&\dot{x}_n(t) = [f_n(t) - A(r_n)x_n(t)]^0 \text{ a.e.}\\
\Rightarrow\, &\dot{x}_n(t) = f_n(t) - A(r_n)^0 x_n(t) \text{ a.e.}
\end{aligned}
$$

But from hypothesis $H(A)_1$, we know that there exists $M_2 > 0$ s.t. for all $n \geqslant 1$ and all $t \in T$, we have $\| A(r_n)^0 x_n(t) \| \leqslant M_2$. Hence we have:

$$
\begin{aligned}
\| \dot{x}_n(t) \| &\leqslant \| f_n(t) \| + M_2\\
&\leqslant a_B(t) + b_B(t)M_1 + M_2 = h_B(t) \text{ a.e.}
\end{aligned}
$$

with $h_B(\cdot) \in L^2_+$ and $B = \{r_n, r\}_{n \geqslant 1} \subseteq E$ (note that the bound $M_1 > 0$ derived in the proof of theorem 3.1 holds for all $x_n(\cdot)$ $n \geqslant 1$, because of hypothesis $H(F)_2$). From this last inequality, we deduce that $\{\dot{x}_n\}_{n \geqslant 1} \subseteq L^2(T, \mathbb{R}^N)$ is relatively sequentially w-compact. Also as in the proof of theorem 3.1, we can get that $\{x_n\}_{n \geqslant 1} \subseteq \hat{p}(V)$, where $V = \{g \in L^2(T, \mathbb{R}^N)\colon \| g(t) \| \leqslant \psi_B(t) = a_B(t) + b_B(t)M_1 \text{ a.e.}\}$. Since $\hat{p}(V)$ is compact in $C(T, \mathbb{R}^N)$ (see the proof of theorem 3.1), we get that $\{x_n\}_{n \geqslant 1} \subseteq C(T, \mathbb{R}^N)$ is relatively compact. Finally note that for $n \geqslant 1$, $\| f_n(t) \| \leqslant \psi_B(t)$ a.e. Hence $\{f_n\}_{n \geqslant 1}$ is relatively sequentially w-compact in $L^2(T, \mathbb{R}^N)$. Thus by passing to an appropriate subsequence if necessary, we may assume that $x_n \xrightarrow{s} x$ in $C(\hat{T}, \mathbb{R}^N)$, $\dot{x}_n \xrightarrow{w} v$ *in* $L^2(T, \mathbb{R}^N)$ and $f_n \xrightarrow{w} f$ in $L^2(T, \mathbb{R}^N)$. Clearly $v = \dot{x}$ on T. Then note that for all $n \geqslant 1$

$$
[x_n, -\dot{x}_n - f_n] \in \operatorname{Gr} \hat{A}(r_n)
$$

and

$$
[x_n, -\dot{x}_n - f_n] \xrightarrow{s \times w} [x, -\dot{x} - f] \text{ in } L^2(T, \mathbb{R}^N) \times L^2(T, \mathbb{R}^N).
$$

Also from lemma 4.1, we have that $\hat{A}(r_n) \to \hat{A}(r)$ in $\mathcal{M}_R(L^2(T, \mathbb{R}^N)) \Rightarrow \operatorname{Gr} \hat{A}(r_n) \xrightarrow{K-M} \operatorname{Gr} \hat{A}(r)$ (see Attouch [1], theorem 3.6.2, p. 365). Hence in the limit as $n \to \infty$, we get

$$
\begin{aligned}
&[x, -\dot{x} - f] \in \operatorname{Gr} \hat{A}(r)\\
&\Rightarrow -\dot{x}(t) \in Ax(t) + f(t) \text{ a.e. on } T
\end{aligned}
$$

and $x(v) = \varphi(r)(v)$ $v \in T_0$ (because of hypothesis $H(\varphi_1)$).

Furthermore because of hypothesis $H(F)_2(2)$ and theorem 4.4 of [16], we have that $f(t) \in F(t, x_t, r)$ a.e. Therefore $x \in P(r)$ and so we have proved that

$$
s\text{-}\overline{\lim}\, P(r_n) \subseteq P(r). \tag{1}
$$

Next let $x \in P(r)$. Then by definition, we have

$$
\begin{aligned}
&-\dot{x}(t) \in A(r)x(t) + f(t) \text{ a.e. on } T\\
&x(v) = \varphi(r)(v) \; v \in T_0
\end{aligned}
$$

with $f\in L^1(T,\mathbb{R}^N)$, $f(t)\in F(t,x_t,r)$ a.e. Let

$$m_n(t)=\operatorname{proj}[f(t);F(t,x_t,r_n)]$$

and

$$u(t,y,r_n)=\operatorname{proj}[m_n(t);F(t,y,r_n)],\, y\in C(T_0,\mathbb{R}^N),$$

where for every $C\in P_{fc}(\mathbb{R}^N)$, $\operatorname{proj}(\cdot;C)$ denotes the metric projection function. From theorem 4.2 of [11], we know that $m_n(\cdot)$ is measurable, while from theorem 3.33, p. 322 of Attouch [1], we have that $(t,y,r)\to u(t,y,r)$ is measurable on t, continuous in (y,r) (i.e. a Caratheodory function), hence jointly measurable. Consider the following differential inclusions $n\geqslant 1$:

$$\left.\begin{aligned} &-\dot{x}_n(t)\in A(r_n)x_n(t)+u(t,(x_n)_t,r_n) \text{ a.e. on } T\\ &x_n(v)=\varphi(r_n)(v)\quad v\in T_0.\end{aligned}\right\}.$$

For each $n\geqslant 1$, we can find a solution $x_n(\cdot)\in C(\hat{T},\mathbb{R}^N)$ of the above problem. From the bounds obtained in the first half of the proof, we know that by passing to a subsequence if necessary, we may assume that $x_n\xrightarrow{s}\hat{x}$ in $C(\hat{T},\mathbb{R}^N)$ and $\dot{x}_n\xrightarrow{w}\dot{\hat{x}}$ in $L^2(T,\mathbb{R}^N)$. Let $\gamma_n(\cdot)$, $\gamma(\cdot)\in L^2(T,\mathbb{R}^N)$ s.t. $\gamma_n(t)\in A(r_n)x_n(t)$ a.e., $\gamma(t)\in A(r)x(t)$ a.e. and $\gamma_n(t)=-\dot{x}_n(t)-u(t,(x_n)_t,r_n)$, $\gamma(t)=-\dot{x}(t)-f(t)$ a.e. Note that $\gamma_n\xrightarrow{w}\hat{\gamma}$ in $L^2(T,\mathbb{R}^N)$ with $\hat{\gamma}(t)=-\dot{\hat{x}}(t)-u(t,\hat{x}_t,r)$ a.e. Then we have:

$$\begin{aligned}(-\dot{x}_n(t)+\dot{x}(t),x(t)-x_n(t))&=(\gamma_n(t)-\gamma(t),x(t)-x_n(t))\\ &\quad+(u(t,(x_n)_t,r_n)-f(t),x(t)-x_n(t))\text{ a.e.}\end{aligned}$$

Recalling that $\operatorname{Gr}\hat{A}(r_n)\xrightarrow{K-M}\operatorname{Gr}\hat{A}(r)$ in $L^2(T,\mathbb{R}^N)\times L^2(T,\mathbb{R}^N)$ (since $\hat{A}(r_n)\to\hat{A}(r)$ in $\mathcal{M}_R(L^2(T,\mathbb{R}^N))$), we can find $\beta_n\in L^2(T,\mathbb{R}^N)$, $\beta_n(t)\in A(r_n)x(t)$ a.e. s.t. $\beta_n\xrightarrow{w}\gamma$ in $L^2(T,\mathbb{R}^N)$. Then we have

$$\begin{aligned}(\gamma_n(t)-\gamma(t),x(t)-x_n(t))&=(\gamma_n(t)-\beta_n(t),x(t)-x_n(t))+(\beta_n(t)-\gamma(t),x(t)-x_n(t))\\ &\leqslant(\beta_n(t)-\gamma(t),x(t)-x_n(t))\text{ a.e.}\end{aligned}$$

the last inequality, being a consequence of the monotonicity of the operator $A(r_n)(\cdot)$. Thus we get:

$$\begin{aligned}\|x_n(t)-x(t)\|^2&\leqslant 2\int_0^t(\beta_n(s)-\gamma(s),x(s)-x_n(s))\,ds\\ &\quad+2\int_0^t(u(s,(x_n)_s,r_n)-f(s),x(s)-x_n(s))\,ds\\ &\leqslant 2\int_0^t(\beta_n(s)-\gamma(s),x(s)-x_n(s))\,ds\\ &\quad+2\int_0^t\|u(s,(x_n)_s,r_n)-f(s)\|\cdot\|x(s)-x_n(s)\|\,ds.\end{aligned}$$

Note that

$$\begin{aligned}\|u(s,(x_n)_s,r_n)-f(s)\|&\leqslant\|u(s,(x_n)_s,r_n)-u(s,x_s,r_n)\|+\|u(s,x_s,r_n)-f(s)\|\\ &\leqslant h(F(s,(x_n)_s,r_n),F(s,x_s,r_n))+h(F(s,x_s,r_n),F(s,x_s,r)).\end{aligned}$$

Hence we have:

$$\|x_n(t)-x(t)\|^2 \leqslant 2\int_0^t (\beta_n(s)-\gamma(s), x(s)-x_n(s))\,ds$$
$$+2\int_0^t h(F(s,(x_n)_s,r_n), F(s,x_s,r_n))\cdot\|x(s)-x_n(s)\|\,ds$$
$$+2\int_0^t h(F(s,x_s,r_n), F(s,x_s,r))\cdot\|x(s)-x_n(s)\|\,ds.$$

Recalling that $\beta_n \xrightarrow{w} \gamma$ in $L^2(T,\mathbb{R}^N)$ and $x_n \xrightarrow{s} \hat{x}$ in $C(\hat{T},\mathbb{R}^N)$, and using hypotheses $H(F)_2(2)$ and (3), in the limit as $n\to\infty$, we get

$$\|\hat{x}(t)-x(t)\|^2 \leqslant 2\int_0^t \eta(s)\|\hat{x}_s-x_s\|_\infty\cdot\|x(s)-\hat{x}(s)\|\,ds$$
$$\leqslant 2\int_0^t \eta(s)\|\hat{x}_s-x_s\|_\infty^2\,ds.$$

Set $\theta(s)=\|\hat{x}_s-x_s\|_\infty^2$. Then we have

$$\theta(t)\leqslant 2\int_0^t \eta(s)\theta(s)\,ds.$$

Invoking Gronwall's inequality, we deduce that $\theta(t)=0$ for all $t\in T \Rightarrow \hat{x}=x\in C(\hat{T},\mathbb{R}^N)$. But note that $x_n \in P(r_n) n\geqslant 1$ and $x_n \xrightarrow{s} x$ in $C(T,\mathbb{R}^N)$. Therefore

$$P(r)\subseteq s\text{-}\underline{\lim}\,P(r_n). \tag{2}$$

From (1) and (2) above we get that

$$P(r_n)\xrightarrow{K} P(r) \text{ as } n\to\infty.$$

But $P(r_n)\subseteq \hat{p}(V)$ $n\geqslant 1$ and the latter is compact in $C(\hat{T},\mathbb{R}^N)$. So lemma 4.2 tells us that $P(r_n)\xrightarrow{h} P(r)$. Therefore $P(\cdot)$ is continuous for both the Vietoris and Hausdorff metric topologies as claimed by the theorem. **Q.E.D.**

5. Relaxation

In § 3 we established the existence of solutions for both the "convex" and "nonconvex" problems. In this section, we show that under some additional regularity hypotheses on the orientor field $F(t,y)$, we can in fact show that the solutions of the nonconvex problem are dense in the $C(\hat{T},\mathbb{R}^N)$-topology to those of the "convex" problem. Such a result is usually called in the literature "relaxation theorem". In optimal control, the "relaxed" (i.e. convexified) problem, plays an important role, because on the one hand captures the asymptotic behavior of the minimizing sequences of the original problem and on the other hand, thanks to its convex structure, always has a solution under very general hypotheses on the data (see Avgerinos-Papageorgiou [3], [4], Warga [21] and references therein).

So we consider problem $(*)$ and its "convexified" version:

$$\left.\begin{aligned} &-\dot{x}\in Ax(t)+\overline{\mathrm{conv}}\,F(t,x_t)\text{a.e. on } T\\ &x(v)=\varphi(v), v\in T_0.\end{aligned}\right\}. \quad (*)_c$$

Denote the solution set of $(*)$ by $P(\varphi)$ and that of $(*)_c$ by $P_c(\varphi)$.

We will need the following hypothesis on the orientor field $F(t,y)$.

<u>$H(F)_3$:</u> $F\colon TxC(T_0,\mathbb{R}^N)\to P_f(\mathbb{R}^N)$ is a multifunction s.t.
(1) $t\to F(t,y)$ is measurable,
(2) $h(F(t,y),F(t,z))\leqslant \eta(t)\|y-z\|_\infty$ a.e. with $\eta(\cdot)\in L^1_+$,
(3) $|F(t,y)|\leqslant a(t)+b(t)\|y\|_\infty$ a.e. with $a(\cdot),b(\cdot)\in L^1_+$.

Theorem 5.1. *If hypotheses $H(A)$, $H(F)_3$ and $H(\varphi)$ hold, then $\overline{P(\varphi)}=P_c(\varphi)$ in $C(\hat{T},\mathbb{R}^N)$.*

Proof. From theorem 3.2 we know that $P(\varphi)\neq\emptyset$ and so $P_c(\varphi)\neq\emptyset$. Furthermore, from theorem 3.1 we have that $P_c(\varphi)$ is compact in $C(\hat{T},\mathbb{R}^N)$.

Let $x(\cdot)\in P_c(\varphi)$. Then by definition, we have

$$-\dot{x}(t)\in Ax(t)+f(t)\text{a.e. on } T$$

$$x(v)=\varphi(v), v\in T_0$$

with $f\in L^1(T,\mathbb{R}^N)$, $f(t)\in F(t,x_t)$a.e.

Recall that the map $p\colon L^1(T,\mathbb{R}^N)\to C(\hat{T},\mathbb{R}^N)$, which to each $g\in L^1(T,\mathbb{R}^N)$ assigns the unique solution of the initial value problem $-\dot{x}(t)\in Ax(t)+g(t)$a.e., $x(0)=\varphi(0)$, is sequentially continuous from $L^1(T,\mathbb{R}^N)$ with the weak topology into $C(T,\mathbb{R}^N)$ with the strong topology. As before $\hat{p}\colon L^1(T,\mathbb{R}^N)\to C(\hat{T},\mathbb{R}^N)$ is defined by $\hat{p}(f)(t)=p(f)(t)$ $t\in T$ and $\hat{p}(f)(v)=\varphi(v)$ $v\in T_0$. Let $V\subseteq L^1(T,\mathbb{R}^N)$ be as in the proof of theorem 3.1. Then V equipped with the relative weak-$L^1(T,\mathbb{R}^N)$ topology, is compact, metrizable. So $\hat{p}\colon V\to C(\hat{T},\mathbb{R}^N)$ is "weak-to-strong" continuous. Thus given $\varepsilon>0$, we can find a symmetric, weak neighborhood of the origin in $C(\hat{T},\mathbb{R}^N)$s.t. if $f-f_1\in U\cap V$, then $\|x-\hat{p}(f_1)\|=\|x-x_1\|_\infty<\varepsilon$ (here we have set $x_1=\hat{p}(f_1)$). From theorem 4.2 of [17], we know that we can choose $f_1\in S^1_{F(\cdot,x.)}$. Next, via a straightforward application of Aumann's selection theorem (see Wagner [20], theorem 5.10), we can find $f_2\in S^1_{F(\cdot,x_1.)}$ s.t.

$$d(f_1(t),F(t,(x_1)_t))=\|f_1(t)-f_2(t)\|\text{a.e.}$$

$$\Rightarrow\|f_1(t)-f_2(t)\|\leqslant h(F(t,x_t),F(t,(x_1)_t))\leqslant\eta(t)\|x_t-(x_1)_t\|_\infty<\eta(t)\varepsilon\text{a.e.}$$

Suppose $f_1,\dots,f_n\in L^1(T,\mathbb{R}^N)$ have been chosen

$$f_{k+1}(t)\in F(t,(x_k)_t)\text{a.e.}\, k=0,1,\dots,n-1\,(x_0=x) \tag{3}$$

$$x_k=\hat{p}(f_k)\text{ and }\|f_k(t)-f_{k+1}(t)\|\leqslant\frac{\varepsilon\eta(t)}{(k-1)!}\left[\int_0^t\eta(s)ds\right]^{k-1}\text{a.e.} \tag{4}$$

Again through Aumann's selection theorem, we can choose $f_{n+1}\in S^1_{F(\cdot,(x_n).)}$s.t.

$$\|f_{n+1}(t)-f_n(t)\|=d(f_n(t),F(t,x_n)_t))\text{a.e.}$$

$$\leqslant h(F(t,(x_{n-1})_t),F(t,(x_n)_t))\leqslant\eta(t)\|(x_n)_t-(x_{n-1})_t\|_\infty\text{a.e.}$$

But from corollary, 1.7.1, p. 35 of Vrabie [19] (Benilan's inequality), we have that

$$\|(x_n)_t-(x_{n-1})_t\|_\infty \leqslant \int_0^t \|f_n(s)-f_{n-1}(s)\|\,\mathrm{d}s \leqslant \int_0^t \frac{\varepsilon\eta(s)}{(n-2)!}\left[\int_0^s \eta(\tau)\mathrm{d}\tau\right]^{n-2}\mathrm{d}s$$

$$\leqslant \int_0^t \frac{\varepsilon}{(n-1)!}\,d\left[\int_0^s \eta(\tau)d\tau\right]^{n-1}$$

$$=\frac{\varepsilon}{(n-1)!}\left[\int_0^t \eta(s)\mathrm{d}s\right]^{n-1}$$

$$\Rightarrow \|f_{n+1}(t)-f_n(t)\| \leqslant \frac{\varepsilon\eta(t)}{(n-1)!}\left[\int_0^t \eta(s)\mathrm{d}s\right]^{n-1} \quad \text{a.e.}$$

Thus by induction, we get a sequence $\{f_k\}_{k\geqslant 1}\subseteq L^1(T,\mathbb{R}^N)$ satisfying (3) and (4) above. Clearly then $\{f_n\}_{n\geqslant 1}\subseteq L^1(T,\mathbb{R}^n)$ is Cauchy. So $f_n \xrightarrow{s} f$ in $L^1(T,\mathbb{R}^N)$. Then $x_n=\hat{p}(f_n)\xrightarrow{s} p(\hat{f})=\hat{x}$ in $C(\hat{T},\mathbb{R}^N)$, and from theorem 4.5 of [16] we have that $\hat{f}(t)\in F(t,\hat{x}_t)$ a.e. Thus $\hat{x}=\hat{p}(\hat{f})\in P(\varphi)$. Also exploiting the monotonicity of $A(\cdot)$, we have

$$(\dot{x}_{k+1}(t)-\dot{x}_k(t), x_{k+1}(t)-x_k(t)) \leqslant (f_{k+1}(t)-f_k(t), x_{k+1}(t)-x_k(t))\ \text{a.e.}$$

$$\Rightarrow \frac{1}{2}\|x_{k+1}(t)-x_k(t)\|^2 \leqslant \int_0^t \|f_{k+1}(s)-f_k(s)\|\cdot\|x_{k+1}(s)-x_k(s)\|\,\mathrm{d}s$$

$$\leqslant \int_0^t \frac{\varepsilon\eta(s)}{(k-1)!}\left(\int_0^s \eta(\tau)\mathrm{d}\tau\right)^{k-1}\|x_{k+1}(s)-x_k(s)\|\,\mathrm{d}s.$$

Invoking lemma A.5, p. 157 of Brezis [6], we get

$$\|x_{k+1}(t)-x_k(t)\| \leqslant \frac{\varepsilon}{k!}\left(\int_0^t \eta(s)\mathrm{d}s\right)^k \quad k\geqslant 1.$$

Hence the triangle inequality gives us

$$\|x_{k+1}(t)-x(t)\| \leqslant \varepsilon\exp\|\eta\|_1$$

$$\Rightarrow \|\hat{x}(t)-x(t)\| \leqslant \varepsilon\exp\|\eta\|_1$$

$$\Rightarrow \|\hat{x}-x\|_\infty \leqslant \varepsilon\exp\|\eta\|_1.$$

Since $\hat{x}\in P(\varphi)$ and $\varepsilon>0$ is arbitrary, we will conclude that $\overline{P(\varphi)}=P_c(\varphi)$, the closure taken in $C(\hat{T},\mathbb{R}^N)$. **Q.E.D.**

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Remark on Gronwall's inequality

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Abstract. Gronwall's inequality has many extensions and analogues among them the discrete one. In this paper we present theorems which look like Gronwall's lemma in the classical propositional calculus.

Keywords. Gronwall's inequality; classical propositional calculus.

1. Introduction

One of the most famous inequalities in the theory of differential equations is Gronwall's inequality. Extended and generalized in many directions (see e.g. [1–3, 5]) this inequality has also discrete version embodied in the following theorem.

Theorem. (*discrete analogue of Gronwall's lemma*).

Let

$$x=\{x_n\}_{n=1}^{\infty},\quad a=\{a_n\}_{n=1}^{\infty}$$

be any real sequences with a – non-negative, c – any real constant. If

h. $$x_{n+1}\leqslant c+\sum_{j=1}^{n}a_jx_j$$

holds for every $n=1,2,\ldots$, then

c. $$x_{n+1}\leqslant(c+a_1x_1)\prod_{j=2}^{n}(1+a_j)$$

for $n=1,2,\ldots$

For understanding the meaning of "look like" Gronwall's lemma, let us see that in the hypothesis *h.* terms of the unknown sequence x appear on both sides of the inequality. In the thesis (consequence) *c.* the terms of x are estimated, bounded by the terms of a,c, and (generally not necessary) the first element x_1. The theorems presented below look similar. The main result of this note is to show that the theorems have their analogues in many branches of mathematics. We construct our theorems in the classical propositional calculus $\mathfrak{Z}_2$. However, some of them can be considered in other languages or in metalanguage. Furthermore it seems that the results can be used as a manner of proving theorems as well, direct and indirect.

2. Preliminaries

We denote by $p, q, s, t, a_1, a_2, \ldots, b_1, b_2, \ldots, c_1, c_2, \ldots, x_1, x_2, \ldots,$ the infinite set of statement variables and by $\sim$, $\wedge$, $\vee$, $\supset$, $\equiv$ connectives in $\mathfrak{Z}_2$. Furthermore we shall use the symbols $\wedge_{j=1}^{n} a_j$ and $\vee_{j=1}^{n} a_j$ to denote generalized conjunction and disjunction i.e. $(a_n \wedge (a_{n-1} \wedge (.. \wedge (a_2 \wedge a_1)..)))$, $(a_n \vee (a_{n-1} \vee (.. \vee (a_2 \vee a_1)..)))$ respectively. We suppose $\wedge_{j=k}^{k} a_j$, $\vee_{j=k}^{k} a_j$ means a_k. Since we use the axioms in the proofs we recall them

a.1 $(p \supset q) \supset [(q \supset s) \supset (p \supset s)]$
a.2 $[p \supset (p \supset q)] \supset (p \supset q)$
a.3 $p \supset (q \supset p)$
a.4 $p \wedge q \supset p$
a.5 $p \wedge q \supset q$
a.6 $(p \supset q) \supset [(p \supset s) \supset (p \supset q \wedge s)]$
a.7 $p \supset p \vee q$
a.8 $q \supset p \vee q$
a.9 $(p \supset s) \supset [(q \supset s) \supset (p \vee q \supset s)]$
a.10 $(p \equiv q) \supset (p \supset q)$
a.11 $(p \equiv q) \supset (q \supset p)$
a.12 $(p \supset q) \supset [(q \supset p) \supset (p \equiv q)]$
a.13 $(\sim q \supset \sim p) \supset (p \supset q)$

The rules of inference are substitution, modus ponens, and the derivable rule hypothetical syllogism use of which is denoted by /, °C, r_{syl} respectively (see e.g. [4 pp. 170]). For making the proofs shorter we apply different statement forms which are known or can be easily inferred from the axioms. In the sequel, whenever we use some known formula of the classical propositional calculus, we note this *f.x*. The formulae we use in the presented proofs are

f.1 $(p \supset q) \supset (s \vee p \supset s \vee q)$
f.2 $(p \supset q) \supset (s \wedge p \supset s \wedge q)$
f.3 $p \supset p$
f.4 $p \wedge q \vee p \wedge s \supset p \wedge (q \vee s)$
f.5 $(p \wedge q) \wedge s \supset p \wedge (q \wedge s)$
f.6 $(p \vee q) \supset [p \vee (s \vee q)]$
f.7 $(p \supset q) \supset [(s \supset t) \supset (s \vee p \supset t \vee q)]$
f.8 $(p \vee q) \vee s \supset (p \vee s) \vee q$
f.9 $p \supset p \wedge p$
f.10 $(p \supset q) \supset [(s \supset t) \supset (p \wedge s \supset q \wedge t)]$

N denotes the set of positive integers.

3. Main results

We start with inequality which is easy to prove and is embodied in the following

$$\forall_{n \in N} a_n \vee \bigvee_{j=1}^{n} b_j \wedge x_j \supset x_{n+1} \forall \Vdash \forall_{n \in N} a_n \vee b_1 \wedge x_1 \supset x_{n+1}.$$

We prove this by applying n-times $a.8$ and r_{syl} and get

1. $$b_1 \wedge x_1 \supset \bigvee_{j=1}^{n} b_j \wedge x_j.$$

Now by f.1 using the rule of substitution

$$\text{f.1 } p/b_1 \wedge x_1, \quad q \Big/ \bigvee_{j=1}^{n} b_j \wedge x_j, \quad s/a_n - 2.$$

we obtain

2. $$\left(b_1 \wedge x_1 \supset \bigvee_{j=1}^{n} b_j \wedge x_j\right) \supset \left(a_n \vee b_1 \wedge x_1 \supset a_n \vee \bigvee_{j=1}^{n} b_j \wedge x_j\right).$$

Hence using modus ponens 2., 1. we have

$$2.{}^{\circ}\mathrm{C}\ 1. - 3.$$

3. $$a_n \vee b_1 \wedge x_1 \supset a_n \vee \bigvee_{j=1}^{n} b_j \wedge x_j.$$

By premises (pr.), 3., and the rule of syllogism we obtain

$$r_{\mathrm{syl}} 3. - \mathrm{pr.} - c.$$

c. $$a_n \vee b_1 \wedge x_1 \supset x_{n+1}.$$

Remark. If we introduce the connective $\subset$ defined by the truth table

p	q	$p \subset q$
T	T	T
T	F	T
F	T	F
F	F	T

then $p \subset q$ is logically equivalent to $q \supset p$, or by writing $q \supset p$ we denote this statement by $p \subset q$. Then the theorem we have proved above can be expressed in the form more familiar for specialists of differential equations and in fact similiar to the inequality considered in the introduction i.e.

If

h.n. $$x_{n+1} \subset a_n \vee \bigvee_{j=1}^{n} b_j \wedge x_j$$

holds (has logical value truth) for every $n \in N$ then

c.n. $$x_{n+1} \subset a_n \vee b_1 \wedge x_1$$

is also true for every $n \in N$.

In such a meaning our theorem ought to be understood. Notice that the statement considered is true if we replace both in the premise (hypothesis) and conclusion $n \in N$ by $n \in Nm := \{1, 2, \ldots, m\}$. It is evident that by f.1 and

we can obtain many similar statements; e.g.

$$\underset{n\in N}{\forall}\left(a_n \vee \bigvee_{j=1}^{n}(b_j \supset x_j) \supset x_{n+1}\right) \Vdash \underset{n\in N}{\forall}(a_n \vee (b_1 \supset x_1) \supset x_{n+1})$$

or using f.2 instead of f.1

$$\underset{n\in N}{\forall}\left(a_n \wedge \bigvee_{j=1}^{n} b_j \vee x_j \supset x_{n+1}\right) \Vdash \underset{n\in N}{\forall}(a_n \wedge (b_1 \vee x_1) \supset x_{n+1})$$

$$\underset{n\in N}{\forall}\left(a_n \wedge \bigvee_{j=1}^{n} b_j \wedge x_j \supset x_{n+1}\right) \Vdash \underset{n\in N}{\forall}(a_n \wedge (b_1 \wedge x_1) \supset x_{n+1}).$$

Regarding axioms a.7 and a.8 we state that the conclusion of the proved statement will be better if many disjuncts appear in the conclusion's antecedent. This leads to the problem of finding the conclusion. The question that whether such a conclusion exists is left open. In the differential equations theory this problem reduces to solving instead inequality respective equation.

Theorem 1. *If*

h.n $\quad a_n \vee \bigvee_{j=1}^{n} b_j \wedge x_j \supset x_{n+1}$

has logical value truth for every $n \in N$ (for some interpretation) then

c.1 $\quad a_1 \vee b_1 \wedge x_1 \supset x_2$

c.n $\quad a_n \vee \left[\bigvee_{j=1}^{n-1} b_{j+1} \wedge a_j \vee \left(\bigwedge_{j=1}^{n} b_j\right) \wedge x_1\right] \supset x_{n+1}$

for $n = 2, 3, \ldots$ is also true.

Proof.

r_{syl} f.3 $p/a_1 \vee b_1 \wedge x_1$ – h.1 – c.1

Therefore c.1 holds. We shall prove that c.2 is true

f.2 $p/a_1 \vee b_1 \wedge x_1,\ q/x_2,\ s/b_2$ °C h.1 – 1.

1. $\quad b_2 \wedge (a_1 \vee b_1 \wedge x_1) \supset b_2 \wedge x_2$

r_{syl} f.4 $p/b_2,\ q/a_1,\ s/b_1 \wedge x_1$ – 1. – 2.

2. $\quad b_2 \wedge a_1 \vee b_2 \wedge (b_1 \wedge x_1) \supset b_2 \wedge x_2$

f.1 $p/(b_2 \wedge b_1) \wedge x_1,\ q/b_2 \wedge (b_1 \wedge x_1),\ s/b_2 \wedge a_1$ °C f.5 $p/b_2,\ q/b_1,\ s/x_1$ – 3.

3. $\quad b_2 \wedge a_1 \vee (b_2 \wedge b_1) \wedge x_1 \supset b_2 \wedge a_1 \vee b_2 \wedge (b_1 \wedge x_1)$

r_{syl} 3. – 2. – 4.

4. $\quad b_2 \wedge a_1 \vee (b_2 \wedge b_1) \wedge x_1 \supset b_2 \wedge x_2$

r_{syl} 4. – a.7 $p/b_2 \wedge x_2,\ q/b_1 \wedge x_1$ – 5.

5. $b_2 \wedge a_1 \vee (b_2 \wedge b_1) \wedge x_1 \supset b_2 \wedge x_2 \vee b_1 \wedge x_1$

f.1 $p/b_2 \wedge a_1 \vee (b_2 \wedge b_1) \wedge x_1,\ \ q/b_2 \wedge x_2 \vee b_1 \wedge x_1,\ \ s/a_2$°C 5. – 6.

6. $a_2 \vee [b_2 \wedge a_1 \vee (b_2 \wedge b_1) \wedge x_1] \supset a_2 \vee [b_2 \wedge x_2 \vee b_1 \wedge x_1]$

r_{syl} 6. – h.2 – c.2

This means that the theorem holds for $n=2$.

Suppose *c.m* is satisfied for $m=1,2,\ldots,k$. We prove that $c.k+1$ holds.

r_{syl} a.7 $p/a_1,\ \ q/b_1 \wedge x_1$ – c.1 – 7.

7. $a_1 \supset x_2$

r_{syl} a.7 $p/a_m,\ \ q\Big/\left[\bigvee_{j=1}^{m-1} b_{j+1} \wedge a_j\right] \vee \left[\bigwedge_{j=1}^{m} b_j\right] \wedge x_1$ – $c.m$ – 8.

8. $a_m \supset x_{m+1} \quad m=2,3,\ldots,k$

f.2 $p/a_m,\ \ q/x_{m+1},\ \ s/b_{m+1}$°C 8. – 9. $m=2,3,\ldots,k$

f.2 $p/a_1,\ \ q/x_2,\ \ s/b_2$°C 7. – 9.

9. $b_{m+1} \wedge a_m \supset b_{m+1} \wedge x_{m+1} \quad m=1,2,\ldots,k$

r_{syl} f.6 $p/a_k,\ \ q\Big/\left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1,\ \ s\Big/\bigvee_{j=1}^{k-1} b_{j+1} \vee a_j$ – $c.k$ – 10.

10. $a_k \vee \left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1 \supset x_{k+1}$

f.2 $p/a_k \vee \left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1,\ \ q/x_{k+1},\ \ s/b_{k+1}$°C 10. – 11.

11. $b_{k+1} \wedge \left[a_k \vee \left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1\right] \supset b_{k+1} \wedge x_{k+1}$

f.1 $p\Big/\left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1,\ \ q/b_{k+1} \wedge \left[\left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1\right],\ \ s/b_{k+1} \wedge a_k$

°C f.5 $p/b_{k+1},\ \ q\Big/\bigwedge_{j=1}^{k} b_j,\ \ s/x_1$ – 12.

12. $b_{k+1} \wedge a_k \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1 \supset b_{k+1} \wedge a_k \vee b_{k+1} \wedge \left[\left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1\right]$

r_{syl} 1.2 – f.4 $p/b_{k+1},\ \ q/a_k,\ \ s\Big/\left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1$ – 11. – 13.

13. $b_{k+1} \wedge a_k \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1 \supset b_{k+1} \wedge x_{k+1}$

r_{syl} 9.$(m=1)$ – a.7 $p/b_2 \wedge x_2,\ \ q/b_1 \wedge x_1$ – 14.

14. $b_2 \wedge a_1 \supset b_2 \wedge x_2 \vee b_1 \wedge x_1$

f.7 $p/b_2 \wedge a_1,\ \ q/b_2 \wedge x_2 \vee b_1 \wedge x_1,\ \ s/b_3 \wedge a_2,\ \ t/b_3 \wedge x_3$°C 14. –

– °C 9.$(m=2)$ – 15.

15. $$b_3 \wedge a_2 \vee b_2 \wedge a_1 \supset b_3 \wedge x_3 \vee \bigvee_{j=1}^{2} b_j \wedge x_j$$

$$\text{f.7}\, p\Big/ \bigvee_{j=1}^{2} b_{j+1} \wedge a_j,\quad q\Big/ \bigvee_{j=1}^{3} b_j \wedge x_j,\quad s/b_4 \wedge a_3,\quad t/b_4 \wedge x_4 \,{}^{\circ}\text{C } 15. -$$

$$- {}^{\circ}\text{C } 9.(m=3) - 16.$$

16. $$\bigvee_{j=1}^{3} b_{j+1} \wedge a_j \supset \bigvee_{j=1}^{4} b_j \wedge x_j.$$

Repeating the above reasoning we get

17. $$\bigvee_{j=1}^{k-1} b_{j+1} \wedge a_j \supset \bigvee_{j=1}^{k} b_j \wedge x_j.$$

Hence

$$\text{f.7}\, p\Big/ \bigvee_{j=1}^{k-1} b_{j+1} \wedge a_j,\quad q\Big/ \bigvee_{j=1}^{k} b_j \wedge x_j,\quad s/b_{k+1} \wedge a_k \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1,$$

$$t/b_{k+1} \wedge x_{k+1} \,{}^{\circ}\text{C } 17. - {}^{\circ}\text{C } 13. - 18.$$

18. $$\left[b_{k+1} \wedge a_k \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1\right] \vee \bigvee_{j=1}^{k-1} b_{j+1} \wedge a_j \supset \bigvee_{j=1}^{k+1} b_j \wedge x_j$$

$$r_{\text{syl}}\ \text{f.8}\, p/b_{k+1} \wedge a_k,\quad q\Big/ \bigvee_{j=1}^{k-1} b_{j+1} \wedge a_j,\quad s\Big/ \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1 - 18. - 19.$$

19. $$\left(\bigvee_{j=1}^{k} b_{j+1} \wedge a_j\right) \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1 \supset \bigvee_{j=1}^{k+1} b_j \wedge x_j$$

$$\text{f.1}\, p\Big/ \left(\bigvee_{j=1}^{k} b_{j+1} \wedge a_j\right) \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1,\quad q\Big/ \bigvee_{j=1}^{k+1} b_j \wedge x_j,\quad s/a_{k+1} \,{}^{\circ}\text{C } 19. - 20.$$

20. $$a_{k+1} \vee \left[\left(\bigvee_{j=1}^{k} b_{j+1} \wedge a_j\right) \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1\right] \supset a_{k+1} \vee \bigvee_{j=1}^{k+1} b_j \wedge x_j$$

$$r_{\text{syl}}\ 20. - h.k+1 - c.k+1.$$

We obtain $c.k+1$ holds so the proof is complete by induction.

Remark. Note that by a.5, a.8, f.1, and f.2

$$a_{k+1} \vee b_1 \wedge x_1 \supset a_{k+1} \vee \left[\left(\bigvee_{j=1}^{k} b_{j+1} \wedge a_j\right) \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1\right].$$

Therefore the statement we have proved at the beginning of this chapter follows from Theorem 1.

In the next theorem we consider a statement wherein the premise instead generalized wedge stands generalized inverted wedge. We start with similar remarks as before.

Theorem 1. *It is evident by a.7 that*

$$\forall_{n\in N} a_n \vee \bigwedge_{j=1}^{n} b_j \vee x_j \supset x_{n+1} \Vdash \forall_{n\in N} a_n \supset x_{n+1}.$$

and e.g.

$$\underset{n\in N}{\forall} a_n \vee \bigwedge_{j=1}^{n} (b_j \supset x_j) \supset x_{n+1} \Vdash \underset{n\in N}{\forall} a_n \supset x_{n+1}.$$

We suppose that the set of premises is semantically consistent. By a.4 we see that if $p \supset x_{n+1}$ then by the rule of syllogism $p \wedge q \supset x_{n+1}$. Therefore the conclusion would be better if less conjunction appears in the antecedent of consequence.

Theorem 2. *If*

h.n $\qquad a_n \vee \bigwedge_{j=1}^{n} (b_j \vee x_j) \supset x_{n+1}$

has logical value truth for every $n \in N$, *then*

c.n $\qquad a_n \vee (b_1 \vee x_1) \supset x_{n+1} \quad n = 1, 2, ..$

is also true.

Proof.

r_{syl} f.3 $p/a_1 \vee (b_1 \vee x_1)$ — h.1 — c.1.

The theorem holds for $n = 1$. Suppose *c.n* is true for $n = 1, 2, \ldots, k$ then the proof of this yields $c.k + 1$.

r_{syl} a.8 $p/a_1, \; q/b_1 \vee x_1$ — h.1 — a.8 $p/b_2, \; q/x_2$ — 1.

1. $\qquad b_1 \vee x_1 \supset b_2 \vee x_2$

f.10 $p/b_1 \vee x_1, \; q/b_2 \vee x_2, \; s/b_1 \vee x_1, \; t/b_1 \vee x_1$ °C 1. — °C f.3 $p/b_1 \vee x_1$ — 2.

2. $\qquad (b_1 \vee x_1) \wedge (b_1 \vee x_1) \supset (b_2 \vee x_2) \wedge (b_1 \vee x_1)$

r_{syl} f.9 $p/b_1 \vee x_1$ — 2. — 3.

3. $\qquad b_1 \vee x_1 \supset (b_2 \vee x_2) \wedge (b_1 \vee x_1)$

f.1 $p/b_1 \vee x_1, \; q/(b_2 \vee x_2) \wedge (b_1 \vee x_1), \; s/a_2$ °C 3. — 4.

4. $\qquad a_2 \vee (b_1 \vee x_1) \supset a_2 \vee (b_2 \vee x_2) \wedge (b_1 \vee x_1)$

r_{syl} 4. — h.2 — c.2

r_{syl} a.8 $p/a_2, \; q/b_1 \vee x_1$ — 4. — h.2 — a.8 $p/b_3, \; q/x_3$ — 5.

5. $\qquad b_1 \vee x_1 \supset b_3 \vee x_3$

f.10 $p/b_1 \vee x_1, \; q/b_3 \vee x_3, \; s/b_1 \vee x_1, \; t/(b_2 \vee x_2) \wedge (b_1 \vee x_1)$

°C 5. — °C 3. — 6.

6. $\qquad (b_1 \vee x_1) \wedge (b_1 \vee x_1) \supset (b_3 \vee x_3) \wedge [(b_2 \vee x_2) \wedge (b_1 \vee x_1)]$

r_{syl} f.9 $p/b_1 \vee x_1$ — 6. — 7.

7. $$b_1 \vee x_1 \supset \bigwedge_{j=1}^{3} b_j \vee x_j.$$

Following this way we obtain

$$b_1 \vee x_1 \supset a_k \vee (b_1 \vee x_1) \supset x_{k+1} \supset b_{k+1} \vee x_{k+1}$$

and similarly as we obtain 3. and 7. applying f.9, f.10 we get

8. $$b_1 \vee x_1 \supset \bigwedge_{j=1}^{k+1} b_j \vee x_j.$$

Now

$$\text{f.1}\ p/b_1 \vee x_1,\ \ q\Big/\bigwedge_{j=1}^{k+1} b_j \vee x_j,\ \ s/a_{k+1}{}^\circ\text{C } 8. - 9.$$

9. $$a_{k+1} \vee (b_1 \vee x_1) \supset a_{k+1} \vee \bigwedge_{j=1}^{k+1} b_j \vee x_j.$$

Hence

$$r_{\text{syl}}\ 9. - \text{h.k} + 1 - \text{c.k} + 1$$

proves that the theorem holds for $k+1$. By the induction argument theorem is proved.

Remark. By a.7 p/a_n, $q/b_1 \vee x_1 - a_n \supset a_n \vee (b_1 \vee x_1)$. Hence by Theorem 2 we get $a_n \supset x_{n+1}$ what we have noticed before this theorem. Applying f.10 we have

$$\text{f.10}\ p/b_2,\ \ q/b_2 \vee x_2,\ \ s/b_1,\ \ t/b_1 \vee x_1{}^\circ\text{C a.7}\ p/b_2,\ \ q/x_2$$
$$-{}^\circ\text{C a.7}\ p/b_1,\ \ q/x_1 - 1.$$

1. $$b_2 \wedge b_1 \supset (b_2 \vee x_2) \wedge (b_1 \vee x_1).$$

Continuing this reasoning we obtain

2. $$\bigwedge_{j=1}^{n} b_j \supset \bigwedge_{j=1}^{n} b_j \vee x_j.$$

From there

$$\text{f.1}\ p\Big/\bigwedge_{j=1}^{n} b_j,\ \ q\Big/\bigwedge_{j=1}^{n} b_j \vee x_j,\ \ s/a_n{}^\circ\text{C } 2. - 3.$$

3. $$a_n \vee \bigwedge_{j=1}^{n} b_j \supset a_n \vee \bigwedge_{j=1}^{n} b_j \vee x_1$$

$$r_{\text{syl}}\ 3. - \text{h.n} - 4.$$

4. $$a_n \vee \bigwedge_{j=1}^{n} b_j \supset x_{n+1}.$$

The conclusion we have just proved is different from the one given in Theorem 2.
In a similar way we can prove

Theorem 3. *If*

h.n $$a_n \wedge \bigwedge_{j=1}^{n} b_j \vee x_j \supset x_{n+1}$$

has logical value truth for every $n \in N$ *then*

c.1 $$a_1 \wedge (b_1 \vee x_1) \supset x_2$$

c.n $$a_n \wedge \left[(b_1 \vee x_1) \wedge \bigwedge_{j=1}^{n-1} (b_{j+1} \vee a_j) \right] \supset x_{n+1}$$

for $n = 2, 3, \ldots$

is also true.

Some of the theorems we can construct in such a manner have their premises which look like hypothesis in the Gronwall inequality while the conclusion has more different forms. We present as an example one such statement

Theorem 4.

$$\bigvee_{n \in N} \bigwedge_{j=1}^{n} b_j \vee \sim x_j \supset x_{n+1} \Vdash \bigvee_{n \in N} x_{n+1} \vee \sim \bigwedge_{j=1}^{n} b_j.$$

We omit the proof of this theorem.

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Proof of some conjectures on the mean-value of Titchmarsh series – III

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Abstract. With some applications in view, the following problem is solved in some special case which is not too special. Let $F(s)=\Sigma_{n=1}^{\infty}a_n\lambda_n^{-s}$ be a generalized Dirichlet series with $1=\lambda_1<\lambda_2<\ldots,\lambda_n\leqslant Dn$, and $\lambda_{n+1}-\lambda_n\geqslant D^{-1}\lambda_{n+1}^{-\alpha}$ where $\alpha>0$ and $D(\geqslant 1)$ are constants. Then subject to analytic continuation and some growth conditions, a lower bound is obtained for $(1/H)\int_0^H|F(it)|^2\,dt$. These results will be applied in other papers to appear later.

Keywords. Titchmarsh series; mean value; lower bounds.

1. Introduction

In the previous papers [1] and [2] with the same title (as the present one) we proved some conjectures made by the second author [4]. In this paper we formulate a new conjecture (which we believe to be true at least in some modified form) and indicate a slight progress towards it.

Conjecture. *Let $1=\mu_1<\mu_2<\ldots$ be any sequence of real numbers with $1/C\leqslant\mu_{n+1}-\mu_n\leqslant C$ where $C(\geqslant 1)$ is an integer constant and $n=1,2,3,\ldots$. Let us form the sequence $1=\lambda_1<\lambda_2<\ldots$ of all possible (distinct) finite power products of $1=\mu_1,\mu_2,\ldots$ with non-negative integral exponents. Let $s=\sigma+it$, $H(\geqslant 10)$ be a real parameter, and $\{a_n\}$ $(n=1,2,3,\ldots)$ with $a_1=1$ be any sequence of complex numbers (possibly depending on H) such that $F(s)=\Sigma_{n=1}^{\infty}a_n\lambda_n^{-s}$ is absolutely convergent at $s=B$ where $B\geqslant 3$ is an integer constant. Suppose that $F(s)$ can be continued analytically in $(\sigma\geqslant 0, 0\leqslant t\leqslant H)$ and that there exist T_1, T_2 with $0\leqslant T_1\leqslant H^{3/4}, H-H^{3/4}\leqslant T_2\leqslant H$ such that for some $K(\geqslant 30)$ there holds*

$$\max_{\sigma\geqslant 0}(|F(\sigma+iT_1)|+|F(\sigma+iT_2)|)\leqslant K.$$

Finally let $\Sigma_{n=1}^{\infty}|a_n|\lambda_n^{-B}\leqslant H^A$ where $A(\geqslant 1)$ is an integer constant. Then there exists a constant $\delta>0$ (depending only on A, B and C) such that for all $H\geqslant H_0(A,B,C)$ there holds

$$\frac{1}{H}\int_0^H|F(it)|^2\,dt\geqslant\frac{1}{2}\sum_{\lambda_n\leqslant H^{\delta}}|a_n|^2, \tag{1}$$

provided that H^{-1} loglog K does not exceed a small positive constant.

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Remark 1. We can strengthen the Conjecture (1) by replacing $\frac{1}{2}$ by a more specific function of H which is asymptotic to 1 as $H \to \infty$.

Remark 2. By the method of [1] we can prove that

$$\frac{1}{H}\int_0^H |F(it)|\,\mathrm{d}t \geqslant \frac{1}{2}. \tag{2}$$

The Remark 1 is also applicable.

Remark 3. Under the condition

$$\sum_{\lambda_n \leqslant X} |a_n| \leqslant D_0(\log X)^R \ (R = H^{\varepsilon},\ D_0 \text{ any constant},\ X \geqslant 30), \tag{3}$$

we can prove (2). Remark 1 is also applicable. For both these results the conditions involving K are unnecessary. For the results mentioned in Remarks 2 and 3 we refer the reader to [1] and [5].

Remark 4. Actually the proof of (1) in [1] goes through without serious problems until we come to a lower bound for

$$\frac{1}{H}\int_0^H \left|\sum_{n \leqslant H^{\delta}} a_n \lambda_n^{-it}\right|^2 \mathrm{d}t.$$

To apply Montgomery–Vaughan theorem we need good lower bounds for $\lambda_{n+1} - \lambda_n$. These are not available in general. But we can work with $\mu_n = (n_0 + n - 1)/n_0$ where $n_0 (\geqslant 2)$ is any integer constant (of course using Montgomery–Vaughan Theorem). Thus in this special case we can prove Conjecture (1). We can also handle $\mu_n = (1+\beta)^{-1}(n+\beta)$ where $\beta (> 0)$ is any real algebraic constant.

Remark 5. We can formulate Conjecture (1) with no conditions involving K, but instead we have to assume condition (3). Remark 1 is also applicable.

Before closing this section we like to make two important remarks. First $\lambda_n \leqslant \mu_n \leqslant Cn$ which is obvious because $\{\lambda_n\}$ contains the subsequence $\{\mu_n\}$. Secondly for $x \geqslant 1$ and $\eta \geqslant 2C + 1$, we have,

$$\begin{aligned}
\sum_{\lambda_n \leqslant x} 1 &\leqslant x^{\eta} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\eta}} \leqslant x^{\eta}\left(1 - \sum_{n=2}^{\infty}{}' \mu_n^{-\eta}\right)^{-1} \\
&\leqslant x^{\eta}\left\{1 - \sum_{n=2}^{\infty}\left(1 + \frac{n-1}{C}\right)^{-\eta}\right\}^{-1} \\
&\leqslant x^{\eta}\left\{1 - \int_0^{\infty}\left(1 + \frac{u}{C}\right)^{-\eta}\mathrm{d}u\right\}^{-1} = x^{\eta}\left(1 - C\int_0^{\infty}\frac{\mathrm{d}u}{(1+u)^{\eta}}\right)^{-1} \\
&= x^{\eta}\left(1 - \frac{C}{\eta - 1}\right)^{-1} \leqslant 2x^{\eta}.
\end{aligned}$$

Hence in (1) the condition $\lambda_n \leqslant H^{\delta}$ is equivalent to a condition of the type $n \leqslant H^{\delta}$ with a different constant $\delta > 0$.

2. Main lemma

Let r be a positive integer, $H \geqslant (r+5)U$ where $U \geqslant 2^{70}(16B)^2$ and M and N are positive integers subject to $N > M \geqslant 1$, and $B(\geqslant 3)$ an integer constant. Let $\{b_m\}$ $(1 \leqslant m \leqslant M)$ and $\{c_n\}$ $(n \geqslant N)$ be two sequences of complex numbers, $1 = \lambda_1 < \lambda_2 < \ldots$ be any increasing sequence of real numbers and let $A(s) = \Sigma_{m \leqslant M} b_m \lambda_m^{-s}$. Let $B(s) = \Sigma_{n \geqslant N} c_n \lambda_n^{-s}$ be absolutely convergent for $s = B$ and continuable analytically in $(\sigma \geqslant 0, 0 \leqslant t \leqslant H)$. Write $g(s) = A(-s)B(s)$,

$$G(s) = U^{-r} \int_0^U du_r \ldots \int_0^U du_1 (g(s+i\lambda))$$

where (here and elsewhere) $\lambda = u_1 + \ldots + u_r$. Assume that there exist real numbers T_1 and T_2 with $0 \leqslant T_1 \leqslant U$, $H - U \leqslant T_2 \leqslant H$, such that

$$|g(\sigma + iT_1)| + |g(\sigma + iT_2)| \leqslant \exp\exp\left(\frac{U}{16B}\right).$$

uniformly in $0 \leqslant \sigma \leqslant B$. Let

$$S_1 = \sum_{m \leqslant M, n \geqslant N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^B 2^r \left(U \log \frac{\lambda_n}{\lambda_m}\right)^{-r}$$

and

$$S_2 = \sum_{m \leqslant M, n \geqslant N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^B.$$

Then, we have,

$$\left| \int_{2U}^{H-(r+3)U} G(it)dt \right| \leqslant 2B^2 U^{-10} + 54BU^{-1} \int_0^H |g(it)|dt$$
$$+ (H + 64B^2)S_1 + 16B^2 S_2 \exp\left(-\frac{U}{8B}\right)$$

Remark. This lemma is borrowed from [1] (see pages 2 to 8).

3. Progress towards the conjecture

From now on we assume that $1 = a_1, a_2, a_3, \ldots$ is any sequence of complex numbers. We set $b_m = a_m$ and $c_n = a_n$ and assume that $\Sigma_{n=1}^{\infty} |a_n| \lambda_n^{-B}$ is convergent.

Lemma 1. *We have, with $\bar{A}(s) = \Sigma_{m \leqslant M} a_m \lambda_n^{-s}$,*

$$\int_{2U}^{H-(r+3)U} |\bar{A}(it)|^2 dt \geqslant (H - (r+5)U - 10\lambda_M \Delta(\lambda_M)) \sum_{m \leqslant M} |a_m|^2,$$

where $\Delta(\lambda_M) = \max\limits_{\substack{\mu \neq \nu \\ 1 \leqslant \mu, \nu \leqslant M}} |\lambda_\mu - \lambda_\nu|^{-1}.$

Proof. Follows from Montgomery-Vaughan Theorem (see [3]).

Lemma 2. We have,

$$S_2 \leqslant \lambda_M^{2B}\left(\sum_{n=1}^{\infty} |a_n|\lambda_n^{-B}\right)^2$$

and

$$S_1 \leqslant 2^r \lambda_M^{2B}\left(\sum_{n=1}^{\infty} |a_n|\lambda_n^{-B}\right)^{-2} (U\lambda_N^{-1}(\lambda_N - \lambda_M))^{-r}.$$

Proof. The first inequality is trivial and the second follows from

$$\log\frac{\lambda_N}{\lambda_M} = -\log\left(1-\left(1-\frac{\lambda_M}{\lambda_N}\right)\right) > \frac{\lambda_N - \lambda_M}{\lambda_N}.$$

We now make the following.

Hypothesis. $\{\lambda_n\}$ is any increasing sequence of real numbers satisfying $\lambda_1 = 1$, $\lambda_n \leqslant Dn, \lambda_{n+1} - \lambda_n \geqslant \lambda_{n+1}^{-\alpha} D^{-1}$, where $D(\geqslant 1)$ is an integer constant and α a positive constant. Also we assume that

$$\sum_{n=1}^{\infty} |a_n|\lambda_n^{-B} \leqslant H^{r\varepsilon/8}$$

where $0 < \varepsilon \leqslant 1/[2(\alpha+1)]$ and $r \geqslant [(200B + 200)\varepsilon^{-1}]$ is any integer. Also $F(s) = \Sigma_{n=1}^{\infty} a_n \lambda_n^{-s}$ shall be as in the introduction except that the $\{\lambda_n\}$ are not related to the $\{\mu_n\}$. $\{\lambda_n\}$ will now be an independent sequence.

From now on we set $N = M + 1$, $M = [H^{(1/(\alpha+1)-\varepsilon)}]$, $U = H^{1-(\varepsilon/2)} + 50B \log\log K_1$ where $K_1 = H^r K$. Note that if $H \geqslant (r+5)U$ is not satisfied, our main theorem (to follow) asserts that a positive quantity is non-negative. Also note that

$$\min_{0\leqslant t\leqslant H^{3/4}} \max_{\sigma\geqslant 0} |F(\sigma+it)|$$

$$\geqslant \min_{0\leqslant t\leqslant U} \max_{\sigma\geqslant 0} |F(\sigma+it)|$$

and a similar result holds for the intervals $(H - H^{3/4}, H)$ and $(H - U, H)$.

Lemma 3. We have,

$$S_2 \leqslant (DH)^{2B} H^{r\varepsilon/4}, S_1 \leqslant 2^r (DH)^{2B} H^{r\varepsilon/4}((2D)^{-\alpha-2} H^{\varepsilon/2})^{-r}$$

and

$$\lambda_M \Delta(\lambda_M) \leqslant D^{\alpha+2} H^{1-\varepsilon}.$$

Proof. We have $\lambda_M \leqslant DM \leqslant DH$ and this proves the first inequality. Also

$$U\lambda_N^{-1}(\lambda_N - \lambda_M) \geqslant H^{1-(\varepsilon/2)}\lambda_N^{-1-\alpha}D^{-1} \geqslant H^{1-(\varepsilon/2)}D^{-1}(DN)^{-1-\alpha}$$
$$\geqslant H^{1-(\varepsilon/2)}(2DM)^{-1-\alpha}D^{-1} \geqslant H^{1-(\varepsilon/2)}(2D)^{-\alpha-2}H^{-1+\varepsilon},$$

and this proves the second inequality. The third follows from

$$\lambda_M \Delta(\lambda_M) \leqslant \lambda_M D \lambda_M^{\alpha} \leqslant D^{\alpha+2} M^{1+\alpha} \leqslant D^{\alpha+2} H^{1-\varepsilon}.$$

The lemma is completely proved.

Now we apply the main lemma (we closely follow the proof of the first main theorem in [1]). Let

$$A(s) = \sum_{m \leqslant M} \bar{a}_m \lambda_m^{-s}, \quad \bar{A}(s) = \sum_{m \leqslant M} a_m \lambda_m^{-s}$$

and

$$B(s) = \sum_{n \geqslant N} a_n \lambda_n^{-s}.$$

Then, we have, in $\sigma \geqslant B$, $F(s) = \bar{A}(s) + B(s)$ and so

$$\begin{aligned} |F(it)|^2 &= |\bar{A}(it)|^2 + 2\,\mathrm{Re}\,(g(it)) + |B(it)|^2 \\ &\geqslant |\bar{A}(it)|^2 + 2\,\mathrm{Re}\,(g(it)), \end{aligned}$$

where $g(s) = A(-s)B(s)$. Hence

$$\begin{aligned} &\int_0^H |F(it)|^2 \, \mathrm{d}t \\ &\geqslant U^{-r} \int_0^U \mathrm{d}u_r \ldots \int_0^U \mathrm{d}u_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} (|\bar{A}(it)|^2 + 2\,\mathrm{Re}\, g(it))\,\mathrm{d}t \\ &= J_1 + 2J_2 \text{ say.} \end{aligned}$$

By Lemmas 1 and 3, we have,

$$J_1 \geqslant (H - (r+5)U - 10D^{\alpha+2} H^{1-\varepsilon}) \sum_{n \leqslant M} |a_n|^2.$$

Again, we have, for $0 \leqslant \sigma \leqslant B$,

$$\begin{aligned} |g(s)| &= |A(-s)B(s)| = |A(-s)(F(s) - A(s))| \\ &\leqslant \Big(\sum_{n \leqslant M} |a_n| \lambda_n^B \Big) K + \Big(\sum_{n \leqslant M} |a_n| \lambda_n^B \Big)^2 \\ &\leqslant \lambda_M^{2B} H^{r\varepsilon/8} K + \lambda_M^{4B} H^{r\varepsilon/4}. \end{aligned}$$

Hence

$$\begin{aligned} |g(s)|_{t=T_1} + |g(s)|_{t=T_2} &\leqslant 2K \lambda_M^{4B} H^{r\varepsilon/4} (H^{-r\varepsilon/8} \lambda_M^{-2B} + K^{-1}) \\ &\leqslant K(DH)^{4B} H^{r\varepsilon/4} \leqslant H^r K = K_1, \end{aligned}$$

the last two inequalities being true for instance if $H \geqslant 10D$. Observe that

$$\operatorname{expexp}\left(\frac{U}{16B}\right) \geqslant \operatorname{expexp}\left(\frac{50}{16} \operatorname{loglog} K_1\right) \geqslant K_1$$

and hence the condition on g and U required by the main lemma is satisfied. Hence by the main lemma, we have,

$$|J_2| \leqslant \frac{2B^2}{U^{10}} + \frac{54B}{U}\int_0^H |g(it)|\,\mathrm{d}t + (H + 64B^2)S_1 + 16B^2 S_2 \exp\left(-\frac{U}{8B}\right)$$

provided $H \geqslant (r+5)U$ and $U \geqslant 2^{70}(16B)^2$. As remarked already we can ignore the condition $H \geqslant (r+5)U$. Also we will satisfy $H \geqslant (50rBD^{B+\alpha+2})^{8/\varepsilon}$ and we will show later that this implies $U \geqslant 2^{70}(16B)^2$. We can assume that $\int_0^H |F(it)|^2 \mathrm{d}t \leqslant H\sum_{n\leqslant M}|a_n|^2$ (otherwise the result asserted by the main theorem to follow, is trivially true). Hence

$$\int_0^H |g(it)|\,\mathrm{d}t = \int_0^H |A(-it)B(it)|\,\mathrm{d}t$$

$$\leqslant \int_0^H |A(-it)|^2\,\mathrm{d}t + \int_0^H |B(it)|^2\,\mathrm{d}t$$

$$\leqslant 3\int_0^H |A(-it)|^2\,\mathrm{d}t + 2\int_0^H |F(it)|^2\,\mathrm{d}t$$

(on noting that $B(it) = F(it) - \bar{A}(it)$)

$$\leqslant (5H + 10D^{\alpha+2}H^{1-\varepsilon}) \sum_{n\leqslant M} |a_n|^2$$

by Montgomery–Vaughan Theorem and the third part of Lemma 3. Hence

$$2|J_2| \leqslant \frac{4B^2}{H^5} + \frac{108B}{H^{1-(\varepsilon/2)}}(5H + 10D^{\alpha+2}H^{1-\varepsilon}) \sum_{n\leqslant M} |a_n|^2$$

$$+ (2H + 128B^2)S_1 + 32B^2 S_2 \exp\left(-\frac{U}{8B}\right)$$

$$\leqslant \left\{\frac{4B^2}{H^5} + \frac{108B}{H^{1-(\varepsilon/2)}}(5H + 10D^{\alpha+2}H^{1-\varepsilon}) + (2H + 128B^2)S_1 + 32B^2 S_2 \exp\left(-\frac{U}{8B}\right)\right\} \sum_{n\leqslant M} |a_n|^2.$$

Thus

$$\int_0^H |F(it)|^2\,\mathrm{d}t \geqslant (H - S_3) \sum_{n\leqslant M} |a_n|^2,$$

where $S_3 > 0$ and

$$S_3 = (r+5)U + 10D^{\alpha+2}H^{1-\varepsilon} + \frac{4B^2}{H^5} + \frac{108B}{H^{1-\varepsilon/2}}(5H + 10^{\alpha+2}H^{1-\varepsilon})$$

$$+ (2H + 128B^2)2^r(DH)^{2B}(2D)^{r(\alpha+2)}H^{-r\varepsilon/4}$$

$$+ 32B^2(DH)^{2B}H^{r\varepsilon/4}r^r(8B)^r H^{-r(2-\varepsilon)/2}.$$

Here we have used $\exp\left(-\frac{U}{8B}\right) \leqslant \frac{r^r(8B)^r}{U^r}$. Note that

$$\log\log K_1 \leqslant \log\log(H^r K) \leqslant \log\log K + \log(r \log H)$$
$$\leqslant \log\log K + \log r + \log H$$

and that $\frac{1}{2}(\log H)^2 \leqslant H$ and so $\log H \leqslant 2H^{1/2}$ and so

$$\frac{\log H}{H^{1-\varepsilon/4}} \leqslant \frac{2}{H^{(1/2)-(\varepsilon/4)}} \leqslant 2H^{-1/4}.$$

Hence

$$(r+5)U \leqslant 100Br(\log\log K + \log r + \log H) + (r+5)H^{1-\varepsilon/2}.$$

Thus

$$S_3 \leqslant 100Br \log\log K + r(\log r)D^{\alpha+2}H^{1-(\varepsilon/4)}\left\{\frac{100B\log H}{H^{1-(\varepsilon/4)}}\right.$$
$$+\frac{(r+5)}{r(\log r)D^{\alpha+2}H^{\varepsilon/4}}+\frac{10}{r(\log r)H^{3\varepsilon/4}}+\frac{4B^2}{H^5}+\frac{108B(15D^{\alpha+2})H}{r(\log r)D^{\alpha+2}H^{2-(3\varepsilon/4)}}$$
$$+\frac{130B^2H2^rD^BH^{2B}(2D)^{r(\alpha+2)}H^{-r\varepsilon/4}}{r(\log r)D^{\alpha+2}H^{1-(\varepsilon/4)}}$$
$$\left.+\frac{32B^2(DH)^{2B}r^r(8B)^rH^{-r(1-\varepsilon/4)}}{r(\log r)D^{\alpha+2}H^{1-(\varepsilon/4)}}\right\}.$$

Denote the expression in the last curly bracket by S_4. Then we have

$$S_4 \leqslant \frac{200B}{H^{\varepsilon/4}}+\frac{2r}{H^{\varepsilon/4}}+\frac{10}{H^{\varepsilon/4}}+\frac{4B^2}{H^{\varepsilon/4}}+\frac{1620B}{H^{\varepsilon/4}}+\frac{130B^2D^BH^{2B+1+1}}{H^{\varepsilon/4}}\left(\frac{4D^{\alpha+2}}{H^{\varepsilon/4}}\right)^r$$
$$+\frac{32B^2D^{2B}}{H^{\varepsilon/4}}H^{2B+1}\left(\frac{8Br}{H^{1/2}}\right)^r.$$

Let $H^{\varepsilon/8} \geqslant 4D^{\alpha+2}$. We have $H^{r\varepsilon/8} \geqslant H^{((200B+200)\varepsilon^{-1}-1)\varepsilon/8} \geqslant H^{2B+2}$. Let $H^{1/4} \geqslant 8Br$. We have $H^{r/4} \geqslant H^{2B+1}$. Now both $H^{\varepsilon/8} \geqslant 4D^{\alpha+2}$ and $H^{1/4} \geqslant 8Br$ are satisfied if

$$H \geqslant (32BrD^{\alpha+2})^{8/\varepsilon}.$$

Hence under this only condition, we have,

$$S_4 \leqslant (200B + 2r + 10 + 4B^2 + 1620B + 130B^2D^B + 32B^2D^{2B})H^{-\varepsilon/4}$$
$$\leqslant rB^2D^{2B}H^{-\varepsilon/4}(200+2+10+4+1620+130+32)$$
$$\leqslant 2000rB^2D^{2B}H^{-\varepsilon/4} \leqslant 1$$

provided $H \geqslant (2000rB^2D^{2B})^{4/\varepsilon}$. Now this last condition and $H \geqslant (32BrD^{\alpha+2})^{8/\varepsilon}$ are both satisfied if $H \geqslant (50rBD^{B+\alpha+2})^{8/\varepsilon}$. Finally $U > H^{1-\varepsilon/2} \geqslant H^{3/4} \geqslant (50 \times 200B)^{(8/\varepsilon)(3/4)} = (10{,}000B\cdot B)^{6/\varepsilon} \geqslant 2^{13(6/\varepsilon)}(16B)^2\ (B^{24}/(16B)^2) \geqslant 2^{70}(16B)^2$ since $B \geqslant 3$. Collecting, we have proved the following

Main Theorem. *Let $\{\lambda_n\}(n=1,2,3,\ldots)$ with $\lambda_1 = 1$ be any increasing sequence of real numbers with the properties $\lambda_n \leqslant Dn$ and $\lambda_{n+1}-\lambda_n > D^{-1}\lambda_{n+1}^{-\alpha}$ where $\alpha(>0)$ is a constant and $D(\geqslant 1)$ is an integer constant. Let $\{a_n\}(n=1,2,3,\ldots)$ with $a_1 = 1$ be any sequence*

of complex numbers such that $F(s)=\Sigma_{n=1}^{\infty} a_n\lambda_n^{-s}$ is absolutely convergent at $s=B$, where $B(\geqslant 3)$ is an integer constant. Let $0<\varepsilon<(2(1+\alpha))^{-1}$ and let $r(\geqslant[(200B+200)\varepsilon^{-1}])$ be any integer constant. Let

$$\sum_{n=1}^{\infty}|a_n|\lambda_n^{-B}\leqslant H^{(r\varepsilon/8)}.$$

Assume that $F(s)$ possesses an analytic continuation in $(\sigma\geqslant 0, 0\leqslant t\leqslant H)$ and that there exist T_1, T_2 with $0\leqslant T_1\leqslant H^{3/4}$, $H-H^{3/4}\leqslant T_2\leqslant H$ such that for some $K(\geqslant 30)$ there holds

$$|F(\sigma+iT_1)|+|F(\sigma+iT_2)|\leqslant K$$

uniformly in $0\leqslant\sigma\leqslant B$. Let

$$H\geqslant(50rBD^{B+\alpha+2})^{8/\varepsilon}.$$

Then, there holds,

$$\frac{1}{H}\int_0^H|F(it)|^2\mathrm{d}t\geqslant(1-\phi)\sum_{n\leqslant M}|a_n|^2,$$

where

$$M=H^{\theta},\quad \theta=\frac{1}{1+\alpha}-\varepsilon,$$

and

$$\phi=r(\log r)D^{\alpha+2}H^{-\varepsilon/4}+100H^{-1}Br\log\log K.$$

In view of the two closing remarks at the end of § 1 we can be now deduce some corollaries.

COROLLARY 1.

Let $\mu_n=n$. Then the conjecture is true.

Proof. We can take $C=1$, $\alpha=\varepsilon$ and $D=1$.

COROLLARY 2.

Let $n_0(\geqslant 2)$ be an integer constant and $\mu_n=(n_0+n-1)/n_0$. Then the conjecture is true.

Proof. First, since $\{\mu_n\}$ is a subsequence of $\{\lambda_n\}$ is follows that $\lambda_n\leqslant\mu_n\leqslant n$. To apply the main theorem we have to verify that $\lambda_{n+1}-\lambda_n\geqslant D^{-1}\lambda_{n+1}^{-\alpha}$ holds with some constant $\alpha>0$ and $D(\geqslant 1)$ an integer constant. To prove this we observe that we can assume that $\lambda_{n+1}-\lambda_n\leqslant 1$. In this case

$$\lambda_{n+1}-\lambda_n=\frac{m_1\ldots m_k}{n_0^k}-\frac{n_1\ldots n_l}{n_0^l}\geqslant n_0^{-j}$$

where $j=\max(k,l)$. Now $(1+(1/n_0))^k\leqslant\lambda_{n+1}$ and $(1+(1/n_0))^l\leqslant\lambda_n$ and so $j=\max(k,l)\leqslant(\log\lambda_{n+1})(\log((n_0+1)/n_0)^{-1}$. But $\log(n_0+1)/n_0=-\log(1-(1/n_0+1))>1/(n_0+1)$.

Thus $j \leqslant (n_0 + 1)(\log \lambda_{n+1})$ and so

$$n_0^{-j} \geqslant \lambda_{n+1}^{-\alpha} \text{ where } \alpha = (n_0 + 1)\log n_0.$$

Plainly we can take $D = 1$.

COROLLARY 3.

Let $\beta > 0$ be an algebraic constant and $\mu_n = (n + \beta)/(1 + \beta)$. Then the conjecture is true. (The conjecture is also true for the choice $\mu_1 = 1$, $\mu_n = n + \beta - 1$ for $n > 1$).

Proof. As before $\lambda_n \leqslant \mu_n \leqslant (n\beta + 2n)(\beta + 1)^{-1}$. Also considering the norm of $\lambda_{n+1} - \lambda_n$ (in case it is $\neq 0$) we can prove that $\lambda_{n+1} - \lambda_n \geqslant D\lambda_{n+1}^{-\alpha}$. The latter assertion follows similarly.

Post-script. The results of this paper were necessitated by a lot of applications to the zeros of generalized Dirichlet series. All these applications will form the subject matter of our forthcoming paper "On the zeros of a class of generalized Dirichlet series-XI".

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Combinatorial meaning of the coefficients of a Hilbert polynomial

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Abstract. In [1] Abhyankar defines an ideal $I(p,a)$ generated by certain minors of a matrix X, the entries of X being independent indeterminates, and proves that the Hilbert function of $I(p,a)$ coincides with its Hilbert polynomial $F(V)$ and obtains it in the form

$$F(V) = \sum_{D\geqslant 0} (-1)^D F_D(m,p,a)\binom{C-D+V}{V}.$$

He also proves that $F(V)$ is the number of certain "indexed" monomials of degree V in the entries of X and that the coefficients $F_D(m,p,a)$ are non-negative integers and asks for their combinatorial meaning. In this paper we characterize the indexed monomials in terms of certain sets of lattice paths, called frames, and prove that the coefficients $F_D(m,p,a)$ count certain families of such frames.

Keywords. Determinantal ideals; Hilbert polynomial; lattice paths; binomial determinants.

1. Introduction

Let X be an $m(1)$ by $m(2)$ matrix whose entries X_{ij}'s are indeterminates over some field K, and let $K[X]$ be the ring of polynomials in these $m(1)m(2)$ variables over K. Let $I(p+1)$ denote the ideal, in $K[X]$, generated by all $(p+1)$ by $(p+1)$ minors of X. A part of the second fundamental theorem for vector invariants says that $I(p+1)$ is a prime ideal in $K[X]$. In [1], Abhyankar proves two generalizations of this result, the first of which is as follows. Let a be a bivector of length p and bounded by $m=(m(1), m(2))$, i.e. let a be a bisequence $\{a(k,i)\}$ of positive integers such that

$$a(k,1) < a(k,2) < \ldots < a(k,p) \leqslant m(k),$$

for $k=1,2$. Let $I(p,a)$ denote the ideal, in $K[X]$, generated by all $(p+1)$ by $(p+1)$ minors of X, all i by i minors of X whose row numbers are less than $a(1,i)$, $1\leqslant i\leqslant p$ and all j by j minors of X whose column numbers are less than $a(2,j)$, $1\leqslant j\leqslant p$. Clearly, in the case when $a(1,i)=i=a(2,i)$ for all i, $1\leqslant i\leqslant p$, we have $I(p,a)=I(p+1)$. Abhyankar proves that $I(p,a)$ is a homogeneous prime ideal in $K[X]$. He also considers, for each non-negative integer V, a certain finite set, denoted by $\mathrm{mon}(m,p,a,V)$, of "indexed" monomials in the variables X_{ij}'s determined by a and V. He proves that the cardinality, $F(V)$, of this set is a polynomial in V and obtains it in the form

$$F(V) = \sum_{D=0}^{C} (-1)^D F_D\binom{V+C-D}{C-D},$$

where the positive integer C is the "degree" of $F(V)$ in V and the coefficients F_D are non-negative integers given by

$$F_D = \sum \binom{E}{D} H_E,$$

where the sum is over all integers E and H_E is a sum of determinants whose entries are products of binomial coefficients. He also proves that the Hilbert function and also the Hilbert polynomial of $I(p,a)$ is this polynomial $F(V)$. That is, if $K[X]_v$ is the V-th homogeneous component of the homogeneous ring $K[X]$ and $I(p,a)_v = I(p,a) \cap K[X]_v$, then the dimension of $A_v = K[X]_v / I(p,a)_v$ – as a K-vector space – is $F(V)$, for *every non-negative integer* V. Thus he shows that the set $\mathrm{mon}(m,p,a,V)$, of indexed monomials, forms a K-basis of A_v. He also gives a determinantal basis for A_V using certain standard Young bitableaux. See also [4].

In [1], page 309, Remark (9.14), Abhyankar asks whether the non-negative integer F_D is the cardinality of some "natural" family of finite sets parametrized by m,p,a,D.

In this paper we determine one such family. Thus in § 2 below we define, for each bivector a of length p and bounded by m, certain p-tuples of lattice paths – called frames – and prove the relation between indexed monomials and frames in Theorem (2.5). Using this relation we deduce an alternative expression for the cardinality of $\mathrm{mon}(m,p,a,V)$ and obtain, in (48), a new expression for the coefficients F_D, by comparison with Abhyankar's formula $F(V)$. This shows, in particular, that F_0 is the total number of frames corresponding to the bivector a. This latter result is essentially the same as Theorem 1 of Gessel and Viennot [3]. We also note that lattice paths, different from ours, have been associated to the indexed monomials by Abhyankar and Kulkarni [2].

In § 3 we consider two of the many equivalent expressions for F_D given in [1], and these involve certain integer-valued functions $H_E^{(12)}(m,p,a)$ and $H_E^{(22)}(m,p,a)$ defined in (42a,c) below. We prove that these also have combinatorial interpretation. In proving these results, we follow the method used in [1]: to find the cardinality, $|S|$, of a finite set S, first discover a recursive relation, together with initial conditions, satisfied by $|S|$. Then prove that this recursive relation, with initial conditions, uniquely determines a function. Finally, discover a formula which satisfies this recursive relation and initial conditions. Then $|S|$ is given by that formula. Thus Theorem (3.3.2) says that $H_E^{(22)}(m,p,a)$ is the number of frames having E "antinodes" each and then Theorem (3.3.3) says that F_D is the number of frames in each of which D of the antinodes have been "marked".

From this combinatorial meaning of F_D we easily deduce the following result conjectured by Abhyankar [1] and proved by Udpikar [5]: if for some positive integer E, $F_E = 0$, then $F_D = 0$ for all $D \geqslant E$. Also, we characterize the integer $\bar{C}$ such that (see (45) (iii) below)

$$F_D > 0 \Leftrightarrow D \in [0, \bar{C}].$$

Finally, in Theorem (3.3.4) we obtain $H_E^{(12)}(m,p,a)$ in terms of $H_E^{(22)}(m,p,a)$ and show that $H_E^{(12)}(m,p,a)$ is the number of frames in each of which $S-E$ of the "intermediate" points have been "labelled", where S is the integer $C[2]$ in (33) below.

2. Paths, frames and mon

We denote by Z, N, N^* and Q the set of all integers, non-negative integers, positive integers and rational numbers respectively. For any set S we put $|S| =$ the cardinal number of S. For any $A, B \in Z$ we put $[A, B] = \{D \in Z \mid A \leqslant D \leqslant B\}$. Given $p \in N$ and $M \in \{Z, N, N^*\}$, let $M(p)$ denote the set of all maps from $[1, p]$ to M.

Given any $p \in N$, by a bivector a of length p we mean a bisequence

$$a(k,1) < a(k,2) < \ldots < a(k,p) \tag{1}$$

of positive integers for $k = 1,2$, and we write $\mathrm{len}(a) = p$.

Given $m \in N^*(2)$ we say that a bivector a of length p is bounded by m and we write $a \leqslant m$, if $a(k,p) \leqslant m(k)$, $\forall k \in [1,2]$.

Given $p \in N$ and $m \in N^*(2)$, we put

$$\mathrm{vec}[p] = \text{the set of all bivectors of length } p$$

and

$$\mathrm{vec}(m,p) = \{a \in \mathrm{vec}[p] \mid a \leqslant m\}.$$

Given $m \in N(2)$, we denote by $\mathrm{cub}(m)$ the positive integral rectangle bounded by m i.e.,

$$\mathrm{cub}(m) = \{y \in Z(2) \mid 1 \leqslant y(k) \leqslant m(k),\ k = 1,2\}.$$

2.1 *Paths*

Given $m \in N^*(2)$ and $n \in N$, a directed lattice path (briefly, a path) in $\mathrm{cub}(m)$ is a sequence of points

$$w = (P(0), \ldots, P(n))$$

in $\mathrm{cub}(m)$ such that for $i \in [0, n-1]$, if $P(i) = (x, y)$, then $P(i+1)$ is either $(x+1, y)$ or $(x, y-1)$. That is, $P(i)P(i+1)$ is either a 'vertical' or a 'horizontal' unit segment. See figure 1. Also then $P(i)$ are called the vertices of w and w is called a path joining $P(0)$ to $P(n)$. Thus along w the x-coordinates of the vertices steadily increase and the y-coordinates steadily decrease and we may describe this by saying that w is a non-increasing path in the direction from top right to bottom left in $\mathrm{cub}(m)$.

Let $P(x, y)$ be a vertex of a path w. Then P is called a node of w if

$$(x, y+1) \in w \text{ and } (x+1,\ y) \in w,$$

and P is called a right-end point in w if

$$(x, y-1) \in w,$$

and P is called an antinode of w if

$$(x-1, y) \in w \text{ and } (x, y-1) \in w$$

and P is called an intermediate point of w if

$$(x, y-1) \in w \text{ and } (x-1, y) \notin w.$$

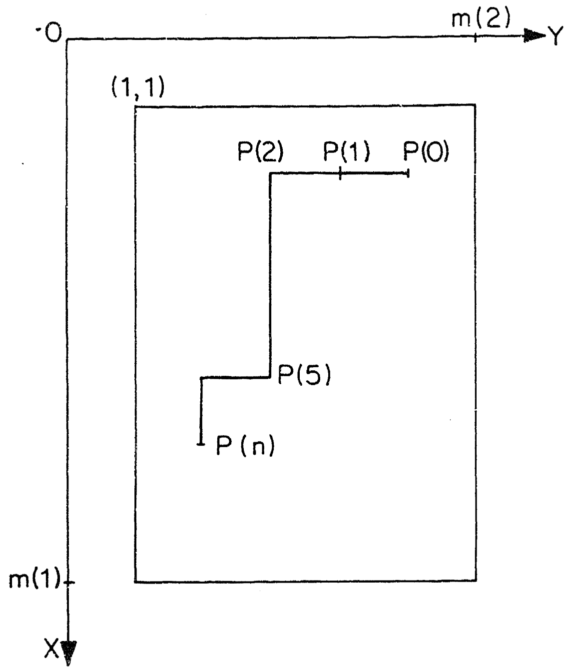

Figure 1.

Thus in figure 1, $P(2)$ is a node, $P(5)$ is an antinode and $P(0)$, $P(1)$ are intermediate points.

It follows from these definitions that

(2) if $P(x, y)$ is a right-end point in a path w, then P is either an antinode or an intermediate point.

Given points $A(a, b)$ and $B(c, d)$ in cub(m) such that

$$a \leqslant c \text{ and } b \geqslant d, \tag{3}$$

let w be the unique path from A to B such that $(c, b) \in w$. We say that this non-empty path w joins A to B minimally or that w is the minimal join of A and B.

(4) Suppose points $A(a, b)$ and $B(c, d)$ satisfy (3) and let $w(1)$ be the path joining A to B minimally. Suppose points $C(a+1,\ b+1)$ and $D(c+1,\ d+1)$ are in cub(m) and are joined minimally by the path $w(2)$. Then clearly $(x, y) \in w(1) \Leftrightarrow (x+1,\ y+1) \in w(2)$. In particular, $w(1)$ and $w(2)$ are non-intersecting.

(5) Given points $A(a, m(2))$ and $B(m(1), b)$ in cub(m), let w be a path from A to B. A point (u, v) in cub(m) is said to lie above the path w if for each vertex (x, y) of w we have either $x > u$ or $y > v$. Also (u, v) is said to lie below w if for each vertex (x, y) of w we have either $x < u$ or $y < v$.

(6) Let the points A, B and path w be as in (5). Then

(i) w has $r = (m(1) - a)$ vertical unit segments, $s = (m(2) - b)$ horizontal unit segments and $r + s + 1$ vertices.

(ii) Suppose a point (u, v) of cub(m) lies below w. Choose vertices $C(c, v)$ and $D(u, d)$ of w. Then the part w' of w, joining C to D, has at least one node (x, y) such that $x < u$ and $y < v$.

Proof. (i) is obvious. For (ii), by data we have $c<u$ and $d<v$. Now since $(u,v)\notin w'$, we see that if x is the largest integer such that $(x,v)\in w'$, then $c\leqslant x<u$ and $(x+1,v)\notin w'$. Hence $(x,v-1)\in w'$. Let y be the smallest integer such that $(x,y)\in w'$. Then $d\leqslant y\leqslant v-1$, and $(x,y-1)\notin w'$. Hence $(x+1,y)\in w'$. Thus (x,y) is a node of w' as required. □

(7) Suppose paths w and w' in cub(m) respectively join $A(a,m(2))$ to $B(m(1),b)$ and $C(a',m(2))$ to $D(m(1),b')$ where $a<a'$ and $b<b'$. Suppose every node of w' lies below w. Then every vertex of w' lies below w and w, w' are non-intersecting.

Proof. Follows by noting first that if points P and P' are below w then every vertex of the minimal join of P and P' lies below w and secondly the join of successive nodes is always minimal. □

(8) Let A, B and w be as in (5). Then for each $c\in[b-m(1),m(2)-a]$, there is a unique point (x,y) on w such that $y-x=c$.

Proof. Obvious if $A=B$. So let $A\neq B$ and $n=m(1)-a+m(2)-b$ and

$$w=(A=P(0),\ P(1),\ldots,P(n)=B).$$

Now $P(1)=(x_1,y_1)$ is such that either

$$x_1=a,\ y_1=m(2)-1 \text{ or } x_1=a+1,\ y_1=m(2).$$

In either case,

$$y_1-x_1=m(2)-a-1.$$

Similarly, for each $i\in[0,n]$, $P(i)=(x_i,y_i)$ satisfies

$$y_i-x_i=m(2)-a-i.$$ □

2.2 *Minimal paths*

Given $m\in N^*(2)$, $p\in N^*$ and $a\in\text{vec}(m,p)$, consider the points

$$A_i=(a(1,i),\ m(2)),\ B_i=(m(1),\ a(2,i)),\ \text{for } i\in[1,p]. \tag{9}$$

The bivector a satisfies, by definition, the inequalities

$$1\leqslant a(k,i)<a(k,i+1)\leqslant m(k), \tag{10}$$

for all $i\in[1,p-1]$ and for all $k\in[1,2]$ and hence also

$$m(k)\geqslant a(k,i)\geqslant i, \tag{11}$$

for all $i\in[1,p]$ and for all $k\in[1,2]$.

As a consequence of these we now prove the existence of a particular non-empty path, denoted by $mw(a,1)$, which joins the point A_1 to B_1.

The path $mw(a,1)$ is obtained by joining minimally the successive points of the sequence

$$\{P(1,j)|j\in[0,2p-1]\}$$

where $\forall j \in [0, p-1]$,

$$P(1,j) = (a(1, 1+j) - j,\ m(2) - j)$$

and $\forall j \in [p, 2p-1]$,

$$P(1,j) = (m(1) - 2p + 1 + j,\ a(2, 2p - j) - 2p + 1 + j).$$

It is easy to see using (10) and (11) that $\forall j \in [0, 2p-2]$, the points $P(1,j)$ and $P(1,j+1)$ belong to $\mathrm{cub}(m)$ and can be joined minimally. Hence the path $mw(a,1)$ is well-defined. Note that if $p = 1$, the path $mw(a,1)$ joins A_1 to B_1 minimally.

The path $mw(a,1)$ will be called the first minimal path belonging to the bivector a. See figure 2. We define the other minimal paths belonging to a inductively as follows. Having defined the path $mw(a,i)$ for $1 \leqslant i < p$, we define the path $mw(a, i+1)$ to be the first minimal path belonging to the bivector $b[i] \in \mathrm{vec}(m, p-i)$, where

$$b[i](k,j) = a(k, j+i),\ \forall j \in [1, p-i] \text{ and } \forall k \in [1,2]. \tag{12}$$

Thus corresponding to $a \in \mathrm{vec}(m,p)$, we have in $\mathrm{cub}(m)$, the p-tuple of paths

$$mW[a] = (mw(a,1), \ldots, mw(a,p)), \tag{13}$$

where $mw(a,i)$ is a particular path joining A_i to $B_i \forall i \in [1,p]$, as defined above. For example, the path $mw(a,2)$ is obtained by joining minimally the successive points of the sequence

$$\{P(2,j) \mid j \in [0, 2p-3]\}$$

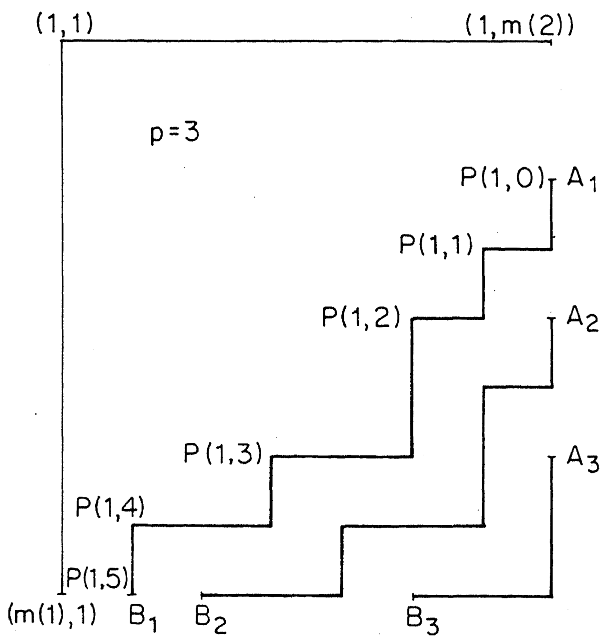

Figure 2.

where for $j \in [0, p-2]$,

$$P(2,j) = (a(1,2+j) - j, m(2) - j)$$

and for $j \in [p-1, 2p-3]$,

$$P(2,j) = (m(1) - 2p + 3 + j,\ a(2, 2p-1-j) - 2p + 3 + j).$$

2.2.1 The above p-tuple $mW[a]$ has the following two properties:

(i) The paths in $mW[a]$ are pairwise non-intersecting.
(ii) Every point of cub(m) lying below the path $mw(a,1)$, lies on a path $mw(a,i)$ for some $i \in [2,p]$.

Proof. Suppose $p > 1$ and consider the paths $mw(a,i)$, $i = 1,2$. In view of (10), we see that $\forall j \in [1, 2p-3]$, the pairs

$$P(1,j),\ P(1,j+1) \text{ and } P(2,j-1),\ P(2,j)$$

satisfy the assumptions of (4). Hence by (4) it follows that
(14) If $(x,y) \in \text{cub}(m)$ with

$$x \geqslant a(1,2) - 1 \text{ and } y < m(2)$$

or

$$x < m(1) \text{ and } y \geqslant a(2,2) - 1$$

then

$$(x,y) \in mw(a,1) \Leftrightarrow (x+1,\ y+1) \in mw(a,2).$$

From this and (10) it follows first that the paths $mw(a,1)$ and $mw(a,2)$ are non-intersecting and so (i) follows by a simple induction on p.

Secondly, (14) can be used to prove (ii). For this note that if $p = 1$ then there is no point of cub(m) lying below $mw(a,1)$ and so there is nothing to prove.

Let $p > 1$. It is enough to show that there is no point of cub(m) which lies below $mw(a,1)$ and above $mw(a,2)$. For then it will follow that if a point P lies below $mw(a,1)$ then P lies on or below $mw(a,2)$ and so (ii) will follow by an easy induction on p.

So let, if possible, $(u,v) \in \text{cub}(m)$ which lies below $mw(a,1)$ and above $mw(a,2)$. Then by (8) there is a point $(x,y) \in mw(a,2)$ such that $y - x = v - u$. Hence $y - v = x - u = j$, say. Now (u,v) is above $mw(a,2)$ and $(x,y) = (u+j,\ v+j) \in mw(a,2)$. Hence either $u + j > u$ or $v + j > v$, so that $j \geqslant 1$. Thus by (14), $(u+j-1,\ v+j-1) \in mw(a,1)$. But (u,v) lies below $mw(a,1)$. Hence either $u + j - 1 < u$ or $v + j - 1 < v$, that is, $j \leqslant 0$. This is a contradiction. □

2.3 *Frames*

Given $m \in N^*(2)$, $p \in N^*$ and $a \in \text{vec}(m,p)$, consider the points A_i and B_i as in (9). By a frame $W[a]$ in cub(m), belonging to the bivector a, we mean an ordered p-tuple

$$W[a] = (w(a,1), \ldots, w(a,p)),$$

where $\forall i \in [1,p]$, $w(a,i)$ is a path in cub(m) joining A_i to B_i and these paths are pairwise non-intersecting. Sometimes we will regard $W[a]$ as a subset of cub(m) containing the vertices of the paths in the frame. This will be clear from the context. Also fr(m,p,a) will denote the set of all frames in cub(m) belonging to a.

Note that, by (2.2.1) (i), the p-tuple $mW[a]$ in (13) is a frame belonging to a and so the set $\mathrm{fr}(m,p,a)$ is non-empty. We call $mW[a]$ the minimal frame belonging to a.
(15) Further, $\forall E\in Z$, let

$$\mathrm{fr}(m,p,a;E) = \{W[a]\in \mathrm{fr}(m,p,a) \mid \text{the } p \text{ paths in } W[a] \text{ have } E \text{ antinodes in all}\}.$$

(16) Note that obviously, $|\mathrm{fr}(m,p,a;E)|=0,\ \forall E<0$.

2.4 *Index and mon*

Let $S\subseteq \mathrm{cub}(m)$ and $e\in N$. By a chain T of length e in S we mean a sequence of points $y_i\in S$, $i\in[1,e]$, such that $y_i(k)<y_{i+1}(k)$, $\forall i\in[1,e-1]$ and $\forall k\in[1,2]$. Also, if T is non-empty, we call the point y_1 the initial point on T.
(17) For every finite subset S of $Z(2)$, we define the index of S, written $\mathrm{ind}(S)$, as follows

$\mathrm{ind}(S)=$ the largest non-negative integer e such that there is a chain of length e in S,

and we note that then $\mathrm{ind}(S)\in N$ and $\mathrm{ind}(S)=0$, if and only if S is empty.

Note that for A, $B\subseteq \mathrm{cub}(m)$, $A\subseteq B \Rightarrow \mathrm{ind}(A)\leqslant \mathrm{ind}(B)$.

Let there be given $p\in N$, $m\in N^*(2)$ and $a\in \mathrm{vec}(m,p)$. Then for every $i\in[1,p]$ we put

$$T(1,i)=\mathrm{truc}(m,p,a,1,i)=\mathrm{cub}(a(1,i)-1,\ m(2)),$$

$$T(2,i)=\mathrm{truc}(m,p,a,2,i)=\mathrm{cub}(m(1),\ a(2,i)-1).$$

$T(k,i)$ will be called the (k,i)-th truncation of $\mathrm{cub}(m)$ with respect to a. Note that by (1) we have

$$T(k,i)\subset T(k,i+1),$$

$\forall i\in[1,p-1]$ and $\forall k\in[1,2]$.

By a *monomial* on $\mathrm{cub}(m)$ we mean a map t of $\mathrm{cub}(m)$ into N. We put

$\mathrm{mon}(m)=$ the set of all monomials on $\mathrm{cub}(m)$,

and for $t\in \mathrm{mon}(m)$ we define the support of t, written $\mathrm{supp}(t)$, and the degree of t, written $\deg(t)$, as follows

$$\mathrm{supp}(t)=\{y\in \mathrm{cub}(m) \mid t(y)\neq 0\},$$

$$\deg(t)=\sum_{y\in \mathrm{cub}(m)} t(y).$$

Also for $V\in N$ we define

$$\mathrm{mon}[[m,V]]=\{t\in \mathrm{mon}(m) \mid \deg(t)=V\}$$

and

$$\mathrm{mon}(m,p)=\{t\in \mathrm{mon}(m) \mid \mathrm{ind}\ (\mathrm{supp}(t))\leqslant p\}$$

and

$$\mathrm{mon}(m,p,a)=\{t\in\mathrm{mon}(m,p)\mid \forall k\in[1,2] \text{ and } \forall i\in[1,p],$$
$$\mathrm{ind}(\mathrm{supp}(t)\cap \mathrm{truc}(m,p,a,k,i))\leqslant i-1\}. \tag{18}$$

Finally, for $V\in N$, we define

$$\mathrm{mon}(m,p,a,V)=\mathrm{mon}(m,p,a)\cap \mathrm{mon}[[m,V]].$$

Let X be an $m(1)$ by $m(2)$ matrix whose entries X_{ij}'s are independent indeterminates over some field K. For every $t\in\mathrm{mon}(m)$, denoted by X^t, the ordinary monomial in X_{ij}'s where

$$X^t=\Pi X_{y(1)y(2)}^{t(y)}$$

where the product is over $y\in\mathrm{cub}(m)$. Note that $t\to X^t$ is a bijection of $\mathrm{mon}(m)$ onto the set of all ordinary monomials in X_{ij}'s, $(i,j)\in\mathrm{cub}(m)$, ana t is thus the "exponent system" of an ordinary monomial. The elements of either of the sets

$$\mathrm{mon}(m,p,a,V) \text{ and } \{X^t\mid t\in\mathrm{mon}(m,p,a,V)\}$$

are called indexed monomials.

Given $S\subseteq\mathrm{cub}(m)$ and $t\in\mathrm{mon}(m)$, we say that S is occupied by t if $\mathrm{supp}(t)\cap S$ is non-empty and that S is free of t otherwise. Given a point P $\mathrm{cub}(m)$, we say that P is occupied by t if $P\in\mathrm{supp}(t)$ and that P is free of t otherwise.

The following two results follow easily from these definitions.

(i) Given $t\in\mathrm{mon}(m)$ and $e\in N^*$ such that $\mathrm{ind}\,(\mathrm{supp}(t))=e$, let (x,y) be the initial point of any chain of length e in $\mathrm{supp}(t)$. Then $\mathrm{cub}\,((x-1,\ y-1))$ is free of t.

(ii) For every $t\in\mathrm{mon}(m,p,a)$, by definition (18) $\forall k\in[1,2]$,

$$\mathrm{ind}(\mathrm{supp}(t)\cap T(k,1))=0.$$

and hence $T(k,1)$ is free of t, $\forall k\in[1,2]$.

2.5 *Relation between mon and frames*

Theorem. *Let a bivector $a\in vec(m,p)$ and a monomial $t\in mon(m)$ be given. Then $t\in mon$ (m,p,a) if and only if there is a frame $W[a]\in fr(m,p,a)$ such that $supp(t)\subseteq W[a]$.*

Proof. First consider any frame $W[a]\in\mathrm{fr}(m,p,a)$ and let $t\in\mathrm{mon}(m)$ be any monomial such that $\mathrm{supp}(t)\subseteq W[a]$.

Suppose

$$W[a]=(w(a,1),\dots,w(a,p)).$$

Given $d\in[1,p]$, let

$$X(d)=\{P\in\mathrm{cub}(m)\mid P\in w(a,i) \text{ for some } i\in[1,d]\}.$$

Then

$$\mathrm{ind}(X(d))\leqslant d. \tag{i}$$

To see this, let S be any (non-empty) chain in $X(d)$ of length e. Then $1 \leqslant e \leqslant d$, because, by the definition of a path, each of the d paths $w(a,1),\ldots,w(a,d)$ can contain at most one point of the chain S. Hence (i) follows.

Now, $w(a,i)$ is a non-increasing path joining the point A_i to B_i, $\forall i \in [1,p]$ and so it follows that for each $i \in [2,p]$, $w(a,1),\ldots,w(a,i-1)$ are exactly the paths from $W[a]$ which meet the truncations $T(k,i)$, $k=1,2$. Therefore, since $\mathrm{supp}(t) \subseteq W[a]$, we see by (i) that $\forall k \in [1,2]$ and $i \in [1,p]$,

$$\mathrm{ind}(\mathrm{supp}(t) \cap T(k,i)) \leqslant i-1$$

and also that

$$\mathrm{ind}(\mathrm{supp}(t)) \leqslant p$$

Hence $t \in \mathrm{mon}(m,p,a)$.

Conversely we prove that

2.5.1: For every $t \in \mathrm{mon}(m,p,a)$, there is a frame $W[a] \in \mathrm{fr}(m,p,a)$ such that $\mathrm{supp}(t) \subseteq W[a]$.

For this we start by constructing for each $t \in \mathrm{mon}(m,p,a)$ a particular path, $wm(a,1,t)$ say, joining point A_1 to B_1 and call this path the first minimal path determined by t. For this construction we shall make use of the minimal frame, $m\,W[a]$, defined in (13).

Given $a \in \mathrm{vec}(m,p)$, for each $i \in [0, m(1)-a(1,1)]$, let $d(i) =$ smallest integer such that $(a(1,1)+i, d(i)) \in mw(a,1)$. Clearly then, $d(i) \geqslant 1$ and $d(i+1) \leqslant d(i)$. Given $t \in \mathrm{mon}(m,p,a)$, for each $i \in [0, m(1)-a(1,1)]$, let $\mathrm{ron}(i,a,t)$ be the integer defined thus:

$$\mathrm{ron}(i,a,t) = \begin{cases} m(2), & \text{if the point } (a(1,1)+i,j) \text{, is free of } t\ \forall j \in [1,m(2)], \\ j, & \text{if } j \text{ is the smallest integer in } [1,m(2)] \text{ such that the point} \\ & (a(1,1)+i,j) \text{ is occupied by } t. \end{cases}$$

Consider the sequence of points

$$\{P(i) = (a(1,1)+i, y(i)) \mid i \in [0, m(1)-a(1,1)]\}$$

where

$$y(0) = \min\{d(0), \mathrm{ron}(0,a,t)\}$$

and for $i \geqslant 1$,

$$y(i) = \min\{y(i-1), d(i), \mathrm{ron}(i,a,t)\}.$$

Then each pair of successive points of the sequence

$$A_1, P(0), \ldots, P(m(1)-a(1,1)), B_1$$

can be joined minimally; let $wm(a,1,t)$ be the path so obtained.

We note the following five properties of this path $wm(a,1,t)$. Put $n = m(1)-a(1,1)$.

(19) Every point of $\mathrm{cub}(m)$, which lies above $wm(a,1,t)$, is free of t.

Proof. This happens because for each point $P = (a(1,1)+i, y)$, $i \in [0,n]$, lying above $wm(a,1,t)$, we have

$$y < y(i) \leqslant \mathrm{ron}(i,a,t). \qquad \square$$

(20) Every vertex of $wm(a, 1, t)$ lies on or above $mw(a, 1)$.

Proof. Follows from the definition of $wm(a, 1, t)$. □

(21) The path $wm(a, 1, t)$ coincides with the path $mw(a, 1)$ if and only if no point above $mw(a, 1)$ is occupied by t.

Proof. If $wm(a, 1, t)$ coincides with $mw(a, 1)$, the result follows from (19). Conversely, suppose every point of cub(m), lying above $mw(a, 1)$ is free of t. Then $y(0) = d(0)$, and $\forall i \in [0, n]$, $d(i) \leqslant \mathrm{ron}(i, a, t)$, and hence $y(i) = d(i)$. □

(22) Every node of $wm(a, 1, t)$, which is not a vertex of $mw(a, 1)$, is necessarily occupied by t.

Proof. Suppose $P = (a(1, 1) + j, y)$, $j \in [0, n-1]$, is a node of $wm(a, 1, t)$ which is not a vertex of $mw(a, 1)$. Then by (20), P lies above $mw(a, 1)$, and so $y < d(j)$. Now, by the definition of $wm(a, 1, t)$, P belongs to the minimal join of the points $P(j-1)$ and $P(j)$. But P is a node and so the points $(a(1, 1) + j,\ y + 1)$ and $(a(1, 1) + j + 1, y)$ belong to $wm(a, 1, t)$. Hence we see that $P = P(j)$ and

$$y = y(j) < y(j - 1).$$

Therefore, $y = y(j) = \mathrm{ron}(j, a, t)$, so that P is occupied by t. □

(23) If for any $i \in [2, p]$,

$$\mathrm{ind}(\mathrm{supp}(t) \cap T(k, i)) = i - 1 \text{ for some } k \in [1, 2], \tag{i}$$

then the initial point of every chain in $\mathrm{supp}(t) \cap T(k, i)$, of length $i - 1$, belongs to $wm(a, 1, t)$. Further, if

$$\mathrm{ind}(\mathrm{supp}(t)) = p, \tag{ii}$$

then the initial point of every chain in supp(t), of length p, belongs to $wm(a, 1, t)$.

Proof. Suppose S is any chain in $\mathrm{supp}(t) \cap T(k, i)$, of length $i - 1$, and let $P = (a(1, 1) + j, y)$ be the initial point of S, where $j \in [0, n]$. If $j = 0$, we see that

$$y(0) \leqslant \mathrm{ron}(0, a, t) \leqslant y$$

because P is occupied by t. Hence $P \in wm(a, 1, t)$. If $j \geqslant 1$, then we will show that

$$y(j) \leqslant y \leqslant y(j - 1). \tag{iii}$$

To prove this, note first that by (i) and (3) we have

$$y \leqslant \mathrm{ron}(d', a, t), \forall d' \in [0, j - 1]. \tag{iv}$$

Secondly, P must lie on or above the path $mw(a, 1)$. For suppose that P lies below $mw(a, 1)$. Then every point of S lies below $mw(a, 1)$. Therefore, by (2.2.1) (ii) it follows that every point of S lies on some path $mw(a, e)$ for $e \in [2, p]$. Also the points of S must belong to distinct paths. But this is impossible because the length of S is $i - 1$, while S must be contained in the union of only the $i - 2$ paths $mw(a, e)$, $e \in [2, i - 1]$, since the paths $mw(a, f)$, $f \in [i, p]$, do not meet the truncations $T(\hat{k}, i)$, $\hat{k} = 1, 2$. Hence

P lies on or above $mw(a,1)$, and so $y \leqslant d(j)$. Therefore, since the sequence d is non-increasing, we also have

$$y \leqslant d(\tilde{e}), \forall \tilde{e} \in [0,j]. \tag{v}$$

Thirdly, we show by induction that, $\forall e \in [0, j-1]$

$$y \leqslant y(e). \tag{vi}$$

For $e = 0$, (vi) follows from (iv), and assuming (vi) for some $e \in [0, h-2]$, we easily obtain from (iv) and (v) that $y \leqslant y(e+1)$. Therefore $y \leqslant y(j-1)$. Finally, note that $y(j) \leqslant \mathrm{ron}(j,a,t) \leqslant y$, because P is occupied. Hence (iii) holds and so $P \in wm(a,1,t)$. Thus (i) is proved. (ii) can be proved similarly. Hence (23) follows.

We define the other minimal paths determined by t inductively, using the bivectors $b[i]$ defined in (12). For example, in order to define the second minimal path $wm(a,2,t)$ for t, consider $t' \in \mathrm{mon}(m)$ thus: $\forall y \in \mathrm{cub}(m)$,

$$(*) \begin{cases} t'(y) = t(y), \text{ if } y \in \mathrm{supp}(t) \setminus wm(a,1,t) \\ \quad\;\; = 0, \text{ otherwise.} \end{cases}$$

Then by (23) we see that

$$t' \in \mathrm{mon}(m, p-1, b[1]).$$

We define $wm(a,2,t)$ to be $wm(b[1], 1, t')$.

These paths $wm(a,i,t)$, $i = 1,2$, are non-intersecting. To see this note that by (19) and by the definition of t', every point of $\mathrm{cub}(m)$, lying on or above $wm(a,1,t)$, is free of t'. Hence every point of $\mathrm{cub}(m)$, occupied by t', lies below $wm(a,1,t)$. Now as in (22), every node P of $wm(a,2,t)$, which is not a vertex of $mw(a,2)$, is necessarily occupied by t'. Hence P lies below $wm(a,1,t)$. Hence by (7), $wm(a,2,t)$ and $wm(a,1,t)$ are non-intersecting.

Repeating the above procedure, we obtain the p-tuple,

$$Wm[a] = (wm(a,1,t), \ldots, wm(a,p,t)),$$

where $\forall i \in [1,p]$, $wm(a,i,t)$ is the i-th minimal path for t. Also these paths are pair-wise non-intersecting and hence $Wm[a] \in \mathrm{fr}(m,p,a)$ and we call $Wm[a]$ the minimal frame for t.

Now we prove the following result which certainly includes (2.5.1).

(2.5.1)′ for every $t \in \mathrm{mon}(m,p,a)$,

$$\mathrm{supp}(t) \subseteq Wm[a],$$

where $Wm[a]$ is the minimal frame for t.

Proof. Induction on p.

Let $p = 1$. Then $a = (a(1,1), a(2,1))$ and the minimal frame $mW[a]$ has only one path $mw(a,1)$, which joins A_1 and B_1 minimally.

Let $t \in \mathrm{mon}(m,p,a)$ be given. If $\mathrm{supp}(t) \subseteq mw(a,1)$, then $mW[a]$ is the required frame by (21). Suppose that some point lying above $mw(a,1)$ is occupied by t. Construct the

path $wm(a,1,t)$. By (19), no point of $\operatorname{supp}(t)$ lies above $wm(a,1,t)$. Let, if possible, $P(u,v)$ be a point lying below $wm(a,1,t)$ and occupied by t. Then, by (6) (ii), $wm(a,1,t)$ has a node $R(x,y)$ (which clearly lies above $mw(a,1)$) such that $x<u$ and $y<v$. Hence by (22), R is occupied by t and so $\{R,P\}$ is a chain of length 2 in $\operatorname{supp}(t)$. Thus $\operatorname{ind}(\operatorname{supp}(t))\geqslant 2$, which contradicts the assumption that $t\in\operatorname{mon}(m,1,a)$. Hence $\operatorname{supp}(t)\subseteq wm(a,1,t)$ and so $Wm[a]$ is the required frame.

Next let $p>1$ and assume the result for all bivectors $a'\in\operatorname{vec}(m,p')$ with $p'\in[1,p-1]$. Let $a\in\operatorname{vec}(m,p)$ be given and let $t\in\operatorname{mon}(m,p,a)$ be also given. Now construct the first minimal path for t, namely $wm(a,1,t)$ and consider the monomial t' as in (*) above. Then, as seen above,

$$t'\in\operatorname{mon}(m,p-1,\,b[1])$$

and so by inductive assumption, $\operatorname{supp}(t')\subseteq Wm[b[1]]$, where

$$Wm[b[1]]=(w_2,\ldots,w_p)$$

is the minimal frame for t'. Now it is easy to see that the frame

$$(wm(a,1,t),w_2,\ldots,w_p)$$

is the same as $Wm[\mathrm{a}]$ and $\operatorname{supp}(t)\subseteq Wm[\mathrm{a}]$. This proves (2.5.1)$'$ and completes the proof of Theorem (2.5). □

(2.6)

For an element V in an overring of Q and $A\in Z$ we define the usual binomial coefficient

$$\binom{V}{A}=\begin{cases}\dfrac{V(V-1)\ldots(V-A+1)}{A!}, & \text{if } A\geqslant 0\\[2mm] 0 & \text{if } A<0\end{cases}$$

and we define the twisted binomial coefficient $\begin{bmatrix}V\\A\end{bmatrix}$ by putting

$$\begin{bmatrix}V\\A\end{bmatrix}=\binom{V+A}{A}.$$

We note that then

(24) if $A\geqslant 0$, each of $\binom{V}{A}$ and $\begin{bmatrix}V\\A\end{bmatrix}$ can be regarded as a polynomial of degree A in an indeterminate V with coefficients in Q and

(25) for V, $A\in N$,

$$\begin{bmatrix}V\\A\end{bmatrix}=\frac{(V+A)!}{V!\,A!}=\begin{bmatrix}A\\V\end{bmatrix}\text{ and}$$

(26) for all A, $V\in Z$,

$$\binom{V}{A}=0,\text{ if } A<0 \text{ or if } A>V\geqslant 0 \text{ and}$$

(27) for all $V \in N$ and $A \in Z$,

$$\binom{V}{A} = \binom{V}{V-A}.$$

For every $S \subseteq \mathrm{cub}\,(m)$ and $V \in N$, put

$$\mathrm{mon}(S) = \{t \in \mathrm{mon}(m) | \mathrm{supp}(t) \subseteq S\}, \tag{28}$$

$$\mathrm{mon}(S, V) = \mathrm{mon}(S) \cap \mathrm{mon}[[m, V]]. \tag{29}$$

Given S, $T \subseteq \mathrm{cub}(m)$ and $V \in N$, it is clear that

$$\mathrm{mon}(S) \cap \mathrm{mon}(T) = \mathrm{mon}(S \cap T), \tag{30}$$

$$\mathrm{mon}(S, V) \cap \mathrm{mon}(T, V) = \mathrm{mon}(S \cap T, V), \tag{31}$$

$$|\mathrm{mon}(S, V)| = \begin{bmatrix} V \\ |S|-1 \end{bmatrix}. \tag{32}$$

Let there be given $p \in N^*$, $m \in N^*$ (2) and $a \in \mathrm{vec}(m, p)$.

Let $W_1, \ldots, W_F$, where $F = |\mathrm{fr}(m, p, a)|$, be all the frames in $\mathrm{fr}(m, p, a)$ labelled in some order. Then by (6) (i) and (2.3) we see that $|W_i| = C + 1$, $\forall i \in [1, F]$, where C is defined as follows.

(33) For every $k \in [1,2]$ let $C[k] = \Sigma_{i=1}^{p}(m(k) - a(k, i))$ and $C = C[1] + C[2] + p - 1$.

Hence by (32) we have

(34) $\forall V \in N$,

$$|\mathrm{mon}(W_i, V)| = \begin{bmatrix} V \\ C \end{bmatrix}, \forall i \in [1, F].$$

(35) Next $\forall r, s \in N^*$, $r \geqslant 2$, let

$$\mathrm{los}(r, s) = \text{family of all sets } \{W_{i_1}, \ldots, W_{i_r}\},\ 1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant F,$$

such that

$$\left| \bigcap_{j=1}^{r} W_{i_j} \right| = C + 1 - s.$$

Then by (31) and (32) we get

(36) $\forall V \in N$ and for every set $\{W_{i_1}, \ldots, W_{i_r}\} \in \mathrm{los}(r, s)$,

$$\left| \bigcap_{j=1}^{r} \mathrm{mon}(W_{i_j}, V) \right| = \left| \mathrm{mon}\left(\bigcap_{j=1}^{r} W_{i_j}, V \right) \right| = \begin{bmatrix} V \\ C-s \end{bmatrix}.$$

In view of (28), by (2.5) we have

$$\mathrm{mon}(m, p, a) = \bigcup_{i=1}^{F} \mathrm{mon}(W_i)$$

and so $\forall V \in N$, by (29),

$$\mathrm{mon}(m, p, a, V) = \bigcup_{i=1}^{F} \mathrm{mon}(W_i, V).$$

Hence by the inclusion–exclusion principle, we get $\forall V \in N$,

$$|\mathrm{mon}(m,p,a,V)| = \sum_{i=1}^{F} \left|\mathrm{mon}(W_i,V)\right| - \sum_{1 \leqslant i<j \leqslant F} |\mathrm{mon}(W_i,V) \cap \mathrm{mon}(W_j,V)|$$

$$+ \sum_{1 \leqslant i<j<l \leqslant F} |\mathrm{mon}(W_i,V) \cap \mathrm{mon}(W_j,V) \cap \mathrm{mon}(W_l,V)| - \ldots$$

$$+(-1)^{F+1}\left|\bigcap_{i=1}^{F} \mathrm{mon}(W_i,V)\right|.$$

Hence by (34) and (31),

$$|\mathrm{mon}(m,p,a,V)| = F\begin{bmatrix} V \\ C \end{bmatrix} - \sum_{1 \leqslant i<j \leqslant F} |\mathrm{mon}(W_i \cap W_j, V)|$$

$$+ \sum_{1 \leqslant i<j<l \leqslant F} |\mathrm{mon}(W_i \cap W_j \cap W_l, V)|$$

$$- \ldots + (-1)^{F+1}\left|\mathrm{mon}\left(\bigcap_{i=1}^{F} W_i, V\right)\right|.$$

Hence by (35) and (36) we get

$$|\mathrm{mon}(m,p,a,V)| = F\begin{bmatrix} V \\ C \end{bmatrix} - \sum_{s \in N^*} |\mathrm{los}(2,s)| \begin{bmatrix} V \\ C-s \end{bmatrix}$$

$$+ \sum_{s \in N^*} |\mathrm{los}(3,s)| \begin{bmatrix} V \\ C-s \end{bmatrix} - \ldots$$

$$+(-1)^{F+1} \sum_{s \in N^*} |\mathrm{los}(F,s)| \begin{bmatrix} V \\ C-s \end{bmatrix} \tag{37}$$

where each sum is clearly essentially finite.

Finally, regrouping the terms in (37) we obtain:

(38) For every $V \in N$,

$$|\mathrm{mon}(m,p,a,V)| = \sum_{D \in N} (-1)^D G_D \begin{bmatrix} V \\ C-D \end{bmatrix}$$

where

$$G_D = \begin{cases} F, \text{ if } D=0 \\ \Sigma_{r \geqslant 2}(-1)^{r+1+D}|\mathrm{los}(r,D)|, \text{ if } D>0. \end{cases} \tag{39}$$

Note that the summation in (39) is essentially finite.

Hence (38) shows that $|\mathrm{mon}(m,p,a,V)|$ is a polynomial in V of degree C (since $G_0 = F > 0$) with coefficients in Q.

Now we describe Abhyankar's formula for $|\mathrm{mon}(m,p,a,V)|$.

(40) Given $p \in N^*$, $D \in Z$ and $M \in \{Z, N\}$, let

$$M(p,D) = \{d \in M(p) \mid d(1) + \ldots + d(p) = D\}.$$

(41) Given $k \in [1,2]$, we put

$$k' = \begin{cases} 2, \text{ if } k=1 \\ 1, \text{ if } k=2 \end{cases}.$$

Let $k\in[1,2]$. First, for $e\in Z(p)$ let $G^{(1k)}(m,p,a,e)$ denote p by p matrix whose (i,j)-th element is

$$G_{ij}^{(1k)}(m,p,a,e)=\binom{m(k)-a(k,j)+j-i}{m(k)-a(k,j)-e(i)}$$

and let

$$G^{[1k]}(m,p,a,e)=\prod_{i=1}^{p}\begin{bmatrix}m(k')-a(k',i)\\ e(i)\end{bmatrix}$$

and

$$H^{(1k)}(m,p,a,e)=G^{[1k]}(m,p,a,e)\det G^{(1k)}(m,p,a,e),$$

where $\det A$ denotes the determinant of the matrix A. Second, for $e\in Z(p)$ let $G^{(1^*k)}(m,p,a,e)$ denote p by p matrix whose (i,j)-th element is

$$G_{ij}^{(1^*k)}(m,p,a,e)=\begin{bmatrix}m(k')-a(k',i)\\ e(j)\end{bmatrix}\binom{m(k)-a(k,j)+j-i}{m(k)-a(k,j)-e(j)}$$

and let

$$H^{(1^*k)}(m,p,a,e)=\det G^{(1^*k)}(m,p,a,e).$$

Third, for every $L\in\{1,1^*\}$ and $E\in Z$ we define

$$H_E^{(Lk)}(m,p,a)=\sum_{e\in Z(p,E)}H^{(Lk)}(m,p,a,e) \tag{42a}$$

and for $D\in Z$ we define

$$F_D^{(Lk)}(m,p,a)=\sum_{E\in Z}H_E^{[Lk]}(m,p,a,D)H_E^{(Lk)}(m,p,a) \tag{42b}$$

where

$$H_E^{[Lk]}(m,p,a,D)=(-1)^{C[k]-E}\binom{E}{D+E-C[k]}.$$

Fourth, for $e\in Z(p)$, let $G^{(2k)}(m,p,a,e)$ and $G^{(2^*k)}(m,p,a,e)$ denote p by p matrices whose (i,j)-th elements are

$$G_{ij}^{(2k)}(m,p,a,e)=\binom{m(k)-a(k,i)+i-j}{m(k)-a(k,i)-e(i)}\begin{bmatrix}m(k')-a(k',j)+j-i\\ e(i)\end{bmatrix}$$

$$G_{ij}^{(2^*k)}(m,p,a,e)=\binom{m(k)-a(k,i)+i-j}{m(k)-a(k,i)-e(j)}\begin{bmatrix}m(k')-a(k',i)+j-i\\ e(j)\end{bmatrix}$$

respectively and let $\forall L\in\{2,2^*\}$

$$H^{(Lk)}(m,p,a,e)=\det G^{(Lk)}(m,p,a,e).$$

Fifth, $\forall L\in\{2,2^*\}$ and $E\in Z$ we define

$$H_E^{(Lk)}(m,p,a)=\sum_{e\in Z(p,E)}H^{(Lk)}(m,p,a,e) \tag{42c}$$

and $\forall L \in \{2, 2^*\}$ and $D \in Z$ we define

$$F_D^{(Lk)}(m,p,a) = \sum_{E \in Z} \binom{E}{D} H_E^{(Lk)}(m,p,a). \tag{42d}$$

Sixth, $\forall L \in \{1, 1^*, 2, 2^*\}$ and $\forall V \in Z$ we define

$$F^{(Lk)}(m,p,a,V) = \sum_{D \in Z} (-1)^D F_D^{(Lk)}(m,p,a) \begin{bmatrix} C-D \\ V \end{bmatrix}. \tag{42}$$

We now quote some results proved by Abhyankar.
(43) ([1], pages 60, 61). For $L \in \{1, 1^*, 2, 2^*\}$ and $k \in [1,2]$, the summations in the definitions (42a) -- (42d) and (42) are essentially finite.
(44) ([1]. page 171). For $L \in \{1, 1^*, 2, 2^*\}$ and $k \in [1,2]$,

$$F_D^{(Lk)}(m,p,a) = F_D^{(11)}(m,p,a), \forall D \in Z.$$

(45) ([1], pages 309, 310). For $L \in \{1, 1^*, 2, 2^*\}$ and $k \in [1,2]$

(i) $F_0^{(Lk)}(m,p,a)$ is a positive integer

(ii) $\forall D \in Z, F_D^{(Lk)}(m,p,a)$ is a non-negative integer

(iii) $\{D \in Z \mid F_D^{(Lk)}(m,p,a) \neq 0\} \subseteq [0, \tilde{C}]$, where

$$\tilde{C} = \min\{C[1], C[2]\},$$

and $C[k]$ are as in (33).

We see, by (43), that the function $F^{(Lk)}(m,p,a,V)$ defined in (42) is a polynomial in V and, by (44), that this polynomial is independent of L and k and so we denote it by $F(V)$. By (45) (i) it follows that the degree of $F(V)$, in V, is the integer C defined in (33).

Finally, we note that ([1], Theorem (9·8), page 306)

$$|\mathrm{mon}(m,p,a,V)| = F(V). \tag{46}$$

In view of (42) and (25), it follows from (46) and (38) above that $\forall L \in \{1, 1^*, 2, 2^*\}$, $\forall k \in \{1,2\}$ and $\forall V \in N$, we have

$$\sum_{D \in N} (-1)^D F_D^{(Lk)}(m,p,a) \begin{bmatrix} V \\ C-D \end{bmatrix} = \sum_{D \in N} (-1)^D G_D \begin{bmatrix} V \\ C-D \end{bmatrix}. \tag{47}$$

Now since $\left\{ \begin{bmatrix} V \\ A \end{bmatrix} \middle| A \in N \right\}$ is a Q-vector space basis of the polynomial ring $Q[V]$, we may compare coefficients of $\begin{bmatrix} V \\ C-D \end{bmatrix}$, $D \in N$, and obtain from (47) that

$$F_D^{(Lk)}(m,p,a) = G_D, \tag{48}$$

$\forall D \in N, \forall L \in \{1, 1^*, 2, 2^*\}$ and $\forall k \in \{1,2\}$.
In particular,

$$F_0 = F = |\mathrm{fr}(m,p,a)|. \tag{49}$$

3. Combinatorial interpretation

3.1 *Notation*

Given $p\in N$ and $A\in Z$, we put

$$J(p)= \text{the set of all subsets of } [1,p] \text{ and}$$
$$J(p,A)=\{a\in J(p)\,|\,|a|=A\}.$$

Given $p\in N^*$, $b\in\text{vec}[p+1]$ and $k\in[1,2]$, with k' as in (41), first for every $u\in J(p)$ we let

$$M[p,b,k,u]=\{a\in\text{vec}[p]\,|\,a(k,i^*)=b(k,i^*)\forall i^*\in[1,p]; \text{ and } a(k',i)\in[b(k',i)+1,\ b(k',i+1)-1], \forall i\in u \text{ and } a(k',j)=b(k',j),\ \forall j\in[1,p]\backslash u.\},$$

and second, $\forall U\in[0,p]$ we let

$$M(p,b,k,U)=\bigcup_{u\in J(p,U)} M[p,b,k,u]. \tag{50}$$

Note that the union is disjoint. Also

$$\forall U\in[0,p],\ M(p,b,k,U)\subset\text{vec}(m,p). \tag{51}$$

Let there be given $k\in[1,2]$ and $m\in N^*(2)$, $m^*\in N^*(2)$ such that
3.1.1: $m^*(k')=m(k')$ and $m^*(k)=m(k)+1$.
Further let $p\in N^*$, $p^*\in N^*$, $a^*\in\text{vec}\ (m^*,p^*)$ and $b\in\text{vec}[p+1]$ be such that either
3.1.2: $p^*=p$ and $m^*(k)-a^*(k,p^*)\neq 0$ and $a^*(\hat{k},i)=b(\hat{k},i)\forall\hat{k}\in[1,2]$ and $i\in[1,p]$ and

$$b(\hat{k},p+1)=m(\hat{k})+1,\forall\hat{k}\in[1,2],$$

holds or
3.1.3: $p^*=p+1, m^*(k)-a^*(k,p^*)=0$ and $a^*=b$
holds.

We note that if (3.1.1) holds and if either (3.1.2) holds or (3.1.3) holds, then $\forall\, U\in[0,p]$, we have

$$M(p,b,k,U)\subset\text{vec}(m,p). \tag{52}$$

3.2 **Theorem.** *Let there be given any $m^*\in N^*(2)$, $p^*\in N^*$, $a^*\in vec(m^*,p^*)$ and $m\in N^*(2)$, $p\in N^*, k\in[1,2]$ and $b\in vec[p+1]$ such that* (3.1.1) *holds and either* (3.1.2) *holds or* (3.1.3) *holds. Then* $\forall E\in Z$,

$$|\text{fr}(m^*,p^*,a^*;\ E)|=\sum_{U\in[0,p]}\ \sum_{a\in M(p,b,k,U)} \text{fr}(m,p,a;\ E-U) \tag{53}$$

and when $p=1$, $m(k)-a(k,1)=0$,

$$|\text{fr}(m,p,a;E)|=\begin{cases}1, \text{ if } E=0\\ 0, \text{ if } E\in Z\backslash\{0\}\end{cases}. \tag{54}$$

Proof. Clearly, by (16), both sides of (53) vanish if $E<0$. Also (54) is obvious.
Let $k=2$ so that $k'=1$; the other case is exactly similar.
Fix $E\in N$ and consider any frame

$$W[a^*]=(w(a^*,1),\dots,w(a^*,p^*))\in \mathrm{fr}(m^*,a^*,p^*;E)$$

where $\forall i\in[1,p^*]$, $w(a^*,i)$ denotes a path joining the points

$$A_i=(a^*(1,i),\ m^*(2)),\quad B_i=(m^*(1),\ a^*(2,i)). \tag{55}$$

$$\left\{\begin{array}{l}\text{Let (3.1.2) hold. Then each path in } W[a^*] \text{ meets the line}\\ Y=m^*(2)-1 \text{ i.e. } Y=m(2) \text{ and } \forall i\in[1,p^*] \text{ we let } A_i' \text{ to be the}\\ \text{vertex with the \textit{smallest} } x \text{ co-ordinate which is common to the}\\ \text{path } w(a^*,i) \text{ and the line } Y=m(2).\\[1ex] \text{Let (3.1.3) hold. Then each path, except } w(a^*,p^*)\text{, in } W[a^*]\\ \text{meets the line } Y=m(2) \text{ and we define the point } A_i' \text{ as before for}\\ \text{every } i\in[1,p^*-1].\end{array}\right. \tag{56}$$

Now let either of (3.1.2) and (3.1.3) hold and define $\forall i\in[1,p]$,

$$\begin{cases} a(1,i)=x \text{ co-ordinate of } A_i' \text{ and}\\ a(2,i)=a^*(2,i),\end{cases} \tag{57}$$

and consider the bisequence

$$a=\{a(\hat{k},i)\,|\,i\in[1,p],\ \hat{k}\in[1,2]\}.$$

Let (3.1.2) hold. Then since the paths in $W[a^*]$ are non-increasing, it follows ($\forall i\in[1,p^*]$) from the definition of the point $A_i'(a(1,i),\ m(2))$ that the point

$$(a(1,i),\ m^*(2))\in w(a^*,i) \text{ and } a(1,i)\geqslant a^*(1,i) \tag{58}$$

and the portion of $w(a^*,i)$ joining $A_i(a^*(1,i),\ m^*(2))$ to A_i' is in fact a minimal join. Further, since the paths in $W[a^*]$ are pair-wise non-intersecting, it follows from (58) that

$$\begin{cases} a(1,i)\in[a^*(1,i),\ a^*(1,i+1)-1],\ \forall i\in[1,p^*-1]\\ \text{and } a(1,p^*)\in[a^*(1,p^*),\ m(1)].\end{cases} \tag{59}$$

Finally, it is clear that $\forall i\in[1,p^*]$, the path $w(a^*,i)$ has an antinode on the line $Y=m^*(2)$ iff

$$a(1,i)>a^*(1,i). \tag{60}$$

If (3.1.3) holds, we see similarly that

$$a(1,i)\in[a^*(1,i),\ a^*(1,i+1)-1],\ \forall i\in[1,p^*-1] \tag{59$'$}$$

and $\forall i\in[1,p^*-1]$, the path $w(a^*,i)$ has an antinode on the line $Y=m^*(2)$ iff (60) holds.
Now let either of (3.1.2) or (3.1.3) hold. Then it follows from (59) or (59)$'$ that for the bisequence a defined by (57), we actually have $a\in\mathrm{vec}(m,p)$. In fact, we see that

there exists a unique integer $U\in[0,p]$ and a unique subset $u\in J(p,U)$ such that

$$a\in M[p,b,2,u],$$

and if $w(a,i)$ denotes the portion of $w(a^*,i)$ joining A_i' to B_i, $\forall i\in[1,p]$, then the p-tuple

$$W[a]=(w(a,1),\ldots,w(a,p)) \tag{61}$$

is, by (60), a frame in $\mathrm{fr}(m,p,a;E-U)$. We shall call $W[a]$, the contraction of $W[a^*]$.

Now consider the set of frames

$$T=\bigcup_{U\in[0,p]}\ \bigcup_{a\in M(p,b,2,U)} \mathrm{fr}(m,p,a;E-U), \tag{62}$$

and note that the unions are disjoint.

We now show that the map

$$W[a^*]\rightarrow W[a]=\text{the contraction of } W[a^*] \tag{63}$$

is a bijection of $\mathrm{fr}(m^*,p^*,a^*,E)$ onto the set T.

To see that the map (63) is surjective, take any $U\in[0,p]$, $u\in J(p,U)$, $a\in M[p,b,2,u]$ and consider a frame $W[a]\in\mathrm{fr}(m,p,a,E-U)$. Let

$$W[a]=(w(a,1),\ldots,w(a,p))$$

where $\forall i\in[1,p]$, $w(a,i)$ is a path joining the point

$$\bar{A}_i(a(1,i),m(2)) \text{ to } \bar{B}_i(m(1),a(2,i)).$$

If (3.1.2) holds, then $p=p^*$ and $a(2,i)=a^*(2,i)$ so that $\bar{B}_i=B_i$, $\forall i\in[1,p^*]$, with A_i, B_i as in (55). Join, $\forall i\in[1,p^*]$, A_i and $\bar{A}_i$ minimally and let $w(a^*,i)$ be the extension of $w(a,i)$ thus obtained, and let

$$W[a^*]=(w(a^*,1),\ldots,w(a^*,p^*)).$$

If (3.1.3) holds, then $p=p^*-1$ and $a^*(2,p^*)=m^*(2)$ and $a(2,i)=a^*(2,i)$ so that $\bar{B}_i=B_i$, $\forall i\in[1,p^*-1]$. For every $i\in[1,p^*-1]$ join A_i and $\bar{A}_i$ minimally. Also join $A_{p^*}(a^*(1,p^*),m^*(2))$ and $B_{p^*}(m^*(1),a^*(2,p^*))$ and let $w(a^*,p^*)$ be the path obtained. Also for every $i\in[1,p^*]$, let $w(a^*,i)$ be the extension of $w(a,i)$ obtained above and let

$$W[a^*]=(w(a^*,1),\ldots,w(a^*,p^*)).$$

Then, if either of (3.1.2), (3.1.3) holds, it is clear that

$$W[a^*]\in\mathrm{fr}(m^*,p^*,a^*;E)$$

and that the contraction of $W[a^*]$ is $W[a]$. Hence the map (63) is surjective.

Next, to show that the map (63) is injective, consider any two *distinct* frames $W_j[a^*]$, $j=1,2$, in $\mathrm{fr}(m^*,p^*,a^*;E)$. Then there exists $i\in[1,p^*]$ such that the paths $w_j(a^*,i)$ in $W_j[a^*]$, $j=1,2$, are different.

Consider the points $A'_{ji}(j=1,2)$ as in (56) and let $W_j[a_j]$ $(j=1,2)$ be the contraction of $W_j[a^*]$. We now show that

$$W_1[a_1] \neq W_2[a_2]. \tag{64}$$

If $A'_{1i} \neq A'_{2i}$, then clearly $a_1 \neq a_2$ and so (64) holds. Hence suppose that $A'_{1i} = A'_{2i}$. Then the portions $w_j(a_j, i)$, of $w_j(a^*, i)$, joining A'_{ji} to $B_{1i}(j=1,2)$ must be different. Hence again (64) holds. Thus the map (63) is injective. Hence Theorem (3.2) is proved. □

(65) *Remark.* It can be easily shown after the manner of [1], lemma (5.6″), page 167, by double induction on p and $s(p) = m(k) - a(k, p)$, that the recurrence relation (53) and the condition (54) uniquely determine a function.

(3.3) In Theorem (3.3.1) below we shall obtain the recurrence relation satisfied by the function $H_E^{(2k)}(m, p, a)$, defined in (42c).

We start by stating the notation and lemmas needed in the proof.

(66) By convention, the sum over an empty family is taken to be zero.

(67) ([1], page 61) For the functions $H_E^{(Lk)}(m, p, a)$ we also have $\forall L \in \{2, 2^*\}$ and $\forall E \in Z$,

$$H_E^{(Lk)}(m, p, a) = \sum_{e \in N(p,E)} H^{(Lk)}(m, p, a, e)$$

where the summation is essentially finite.

Also, by definition, $\forall L \in \{2, 2^*\}$ and $\forall D \in Z$,

$$F_D^{(Lk)}(m, p, a) = \sum_{E \in Z} \binom{E}{D} H_E^{(Lk)}(m, p, a). \tag{68}$$

We note that the summation in (68) is essentially finite.

(69) ([1], page 211) Given $K_{ij}(d^*) \in Q \, \forall i, j = [1, p]$ and $d^* \in Z$. Then $\forall D \in Z$ we have

$$\sum_{d \in N(p,D)} \det[K_{ij}(d_j)] = \sum_{d \in N(p,D)} \det[K_{ij}(d_i)].$$

(70) It follows from (69) that $\forall E \in Z$,

$$H_E^{(2k)}(m, p, a) = H_E^{(2^*k)}(m, p, a).$$

(71) Let $p \in N^*$ be given. For $u \in J(p)$ and $\forall x, y \in Z(p)$ such that $x_i < y_i$, $\forall i \in [1, p]$, we put

$$X[u] = \{r \in Z(p) \mid r_i \in [x_i + 1, y_i - 1], \quad \forall i \in u \text{ and } r_j = y_j \forall j \in [1, p] \backslash u\}$$

and $\forall U \in [0, p]$ we put

$$X(U) = \bigcup_{u \in J(p,U)} X[u] \tag{i}$$

and note that if u', $u^* \in J(p)$ and $u' \neq u^*$ then

$$X[u'] \cap X[u^*] = \varnothing$$

so that
(72) the union in (i) is disjoint.
Hence we see that
(73) Given any $U\in[0,p]$, for every map $f:Z(p)\to Q$, we have

$$\sum_{r\in X(U)} f(r)=\sum_{u\in J(p,U)}\sum_{r\in X[u]} f(r).$$

(74) For the rest of this section we assume that there are given $m^*\in N^*(2)$, $p^*\in N^*$, $a^*\in\mathrm{vec}(m^*,p^*)$ and $m\in N^*$, $p\in N^*$, $k\in[1,2]$ and $b\in\mathrm{vec}[p+1]$ such that (3.1.1) holds and either (3.1.2) holds or (3.1.3) holds.

We shall deal with the case $k=2$ so that $k'=1$; the other case is exactly similar.

Define sequences $r^*,s^*\in Z(p^*)$ by putting $\forall j\in[1,p^*]$,

$$s_j^*=m^*(k)-a^*(k,j),\quad r_j^*=m^*(k')-a^*(k',j). \tag{75}$$

Then since $a^*\in\mathrm{vec}(m^*,p^*)$, it is clear that r^* and s^* are strictly decreasing sequences of non-negative integers.

Let $\forall i\in[1,p]$,

$$x_i=m(k')-b(k',i+1),\quad y_i=m(k')-b(k',i) \tag{76}$$

and

$$s_i=m(k)-b(k,i). \tag{77}$$

Now by assumption $a^*\in\mathrm{vec}(m^*,p^*)$ and (3.1.1) holds. Hence by (76) and (77) we see that
(78) If (3.1.2) holds then first $p=p^*$ and secondly,

$$y_{i+1}=x_i=r_{i+1}^*,\ \forall i\in[1,p-1]$$

and

$$x_p=-1,\ y_1=r_1^*$$

and thirdly, $s_{p^*}^*>0$ and

$$s_i^*-1-p^*+i\geqslant 0,\ \forall i\in[1,p]$$

and

$$s_i=s_i^*-1,\quad \forall i\in[1,p].$$

(79) If (3.1.3) holds then first $p=p^*-1$, and secondly,

$$y_{i+1}=x_i=r_{i+1}^*\ \forall i\in[1,p-1]$$

and

$$x_p=r_{p^*}^*\geqslant 0,\ y_1=r_1^*$$

and thirdly, $s_{p^*}^*=0$ and

$$s_i=s_i^*-1,\ \forall i\in[1,p].$$

Next let

$$\hat{M}=\{a\in\mathrm{vec}(m,p)\,|\,a(k,i)=b(k,i),\ \forall i\in[1,p]\}. \tag{80}$$

and
(81) $\forall a \in \hat{M}$, let $r[a] \in Z(p)$ be defined by putting,

$$\forall i \in [1, p],\ r[a]\ (i) = m(k') - a(k', i).$$

Then, by the definition of M in (3.1) we see that
(82) given $u \in J(p)$ and $a \in \hat{M}$,

$$a \in M[p, b, k, u] \Leftrightarrow r[a](j) \in [x_j + 1, y_j - 1]\ \forall j \in u \\ \text{and } r[a](j) = y_j,\ \forall j \in [1, p] \backslash u$$

and equivalently
(83) given $u \in J(p)$ and $a \in \hat{M}$,

$$a \in M[p, b, k, u]$$

$$\Leftrightarrow \forall j \in u,\ r[a](j) = r_j^* - 1 - t_j \text{ for some } t_j \in [0, r_j^* - r_{j+1}^* - 2] \\ \text{and } r[a](j) = r_j^*,\ \forall j \in [1, p] \backslash u.$$

By (82) we see that $\forall u \in J(p)$

$$a \to r[a]$$

gives a bijection of $M[p, b, k, u]$ onto $X[u]$ when $x, y \in Z(p)$ are chosen as in (76). Hence, in view of (50) and (72), we see that
(84) $\forall U \in [0, p]$,

$$a \to r[a]$$

gives a bijection of $M(p, b, k, U)$ onto $X(U)$.

3.3.1 **Theorem**. *Suppose* (74) *holds. Then* $\forall E \in Z$ *we have*

$$H_E^{(2k)}(m^*, p^*, a^*) = \sum_{U \in [0, p]} \sum_{a \in M(p, b, k, U)} H_{E-U}^{(2k)}(m, p, a)$$

and when

$$p = 1,\ m(2) - a(2, 1) = 0,$$

$$H_E^{(2k)}(m, p, a) = 1, \text{ if } E = 0 \text{ and zero otherwise.}$$

Proof. By definition and (67), we have for $E \in Z$,

$$H_E^{(2k)}(m^*, p^*, a^*) = \sum_{e \in N(p^*, E)} \det\ [G_{ij}(e)]$$

where $\forall i, j \in [1, p^*]$ and $e \in N(p^*, E)$,

$$G_{ij}(e) = \binom{s_i^* + i - j}{s_i^* - e_i} \binom{r_j^* + j - i}{e_i}.$$

Fix $e \in N(p^*, E)$ and consider the matrix $G(e) = [G_{ij}(e)]$. We use the following two basic results

$$\binom{V}{A} - \binom{V-1}{A} = \binom{V-1}{A-1}, \quad \forall A,\ V \in Z, \tag{85}$$

and given any A, V, $W \in Z$ with $V \leqslant W$,

$$\binom{W}{A} - \binom{V}{A} = \sum_{t=0}^{W-V-1} \binom{W-1-t}{A-1}. \tag{86}$$

First let (3.1.2) hold. Then $s^*_{p^*} > 0$.

For every $j \in [1, p^* - 1]$, in $G(e)$ subtract $(j+1)$-th column from j-th column to obtain the matrix $H(e)$ whose (i,j)-th element is, say H_{ij}, $1 \leqslant i \leqslant p^*$ where $H_{ip^*} = G_{ip^*}(e)$ and for $1 \leqslant j < p^*$,

$$H_{ij} = \binom{s^*_i + i - j}{s^*_i - e_i}\binom{r^*_j + j - i}{e_i} - \binom{s^*_i + i - j - 1}{s^*_i - e_i}\binom{r^*_{j+1} + j + 1 - i}{e_i}.$$

Then by (85) we have for $1 \leqslant j < p^*$

$$H_{ij} = \binom{s^*_i + i - j - 1}{s^*_i - e_i}\left\{\binom{r^*_j + j - i}{e_i} - \binom{r^*_{j+1} + j + 1 - i}{e_i}\right\}$$
$$+ \binom{s^*_i + i - j - 1}{s^*_i - e_i - 1}\binom{r^*_j + j - i}{e_i}.$$

and hence by (86),

$$\text{(I)} \qquad H_{ij} = \sum_{t=0}^{r^*_j - r^*_{j+1} - 2} \binom{s^*_i - 1 + i - j}{s^*_i - 1 - (e_i - 1)}\binom{r^*_j - 1 - t + j - i}{e_i - 1}$$
$$+ \binom{s^*_i - 1 + i - j}{s^*_i - 1 - e_i}\binom{r^*_j + j - i}{e_i}.$$

Also, $\forall i \in [1, p^*]$,

$$H_{ip^*} = \binom{s^*_i + i - p^*}{s^*_i - e_i}\binom{r^*_{p^*} + p^* - i}{e_i}$$
$$= \binom{s^*_i + i - p^* - 1}{s^*_i - e_i}\binom{r^*_{p^*} + p^* - i}{e_i} + \binom{s^*_i + i - p^* - 1}{s^*_i - e_i - 1}\binom{r^*_{p^*} + p^* - i}{e_i}$$
$$\text{(II)} \qquad H_{ip^*} = \sum_{t=0}^{r^*_{p^*} - 1} \binom{s^*_i - 1 + i - p^*}{s^*_i - 1 - (e_i - 1)}\binom{r^*_{p^*} - 1 - t + p^* - i}{e_i - 1}$$
$$+ \binom{s^*_i - 1 + i - p^*}{s^*_i - 1 - e_i}\binom{r^*_{p^*} + p^* - i}{e_i}$$
$$+ \binom{s^*_i - 1 + i - p^*}{s^*_i - e_i}\binom{p^* - i}{e_i}, \qquad \text{(by (86))}$$

Now, if $e_i > p^* - i$ then $\binom{p^* - i}{e_i} = 0$ by (26), since $p^* - i \geqslant 0$.

Next let $e_i \leqslant p^* - i$. Then

$$s^*_i - e_i \geqslant s^*_i - p^* + i,$$

and so

$$\binom{s_i^* - 1 - p^* + i}{s_i^* - e_i} = 0$$

by (26), since $s_i^* - 1 - p^* + i \geqslant 0$ by (78). Hence the last term in (II) above is always zero.

Now let (3.1.3) hold. Then $s_{p^*}^* = 0$. Hence $\forall j \in [1, p^*]$,

$$G_{p^*j}(e) = \binom{s_{p^*}^* + p^* - j}{s_{p^*}^* - e_{p^*}} \binom{r_j^* + j - p^*}{e_{p^*}} = \binom{p^* - j}{-e_{p^*}} \binom{r_j^* + j - p^*}{e_{p^*}}.$$

Now $e_{p^*} \geqslant 0$ and $p^* - j \geqslant 0$ $\forall j \in [1, p^*]$.

Hence, by (26), $\forall j \in [1, p^*]$,

$$G_{p^*j}(e) = \begin{cases} 0, & \text{if } e_{p^*} > 0 \\ 1, & \text{if } e_{p^*} = 0. \end{cases}$$

Therefore $\det G(e) = 0$ if $e_{p^*} > 0$ and each element of the last row of $G(e)$ is 1 if $e_{p^*} = 0$. Hence let $e_{p^*} = 0$.

Then $\forall j \in [1, p^* - 1]$ we have

$$H_{p^*j} = 0 \text{ and } H_{p^*p^*} = 1.$$

Expanding $\det H(e)$ by its last row we obtain a determinant of order $p^* - 1 = p$ whose (i, j)-th element is given by (I) above.

Hence, in view of (66), we see by (78) and (79) that when either of (3.1.2) and (3.1.3) holds,

$$\text{(III)} \qquad H_E^{(2k)}(m^*, p^*, a^*) = \sum_{e \in N(p, E)} \det[H_{ij}],$$

where $\forall i, j \in [1, p]$,

$$H_{ij} = \sum_{z_j \in [x_j + 1, y_j - 1]} \binom{s_i + i - j}{s_i - (e_i - 1)} \binom{z_j + j - i}{e_i - 1} + \binom{s_i + i - j}{s_i - e_i} \binom{y_j + j - i}{e_i}.$$

Now for every $u \in J(p)$ and $i \in [1, p]$, let

$$H_{ij}^{(u)} = \begin{cases} \displaystyle\sum_{z_j \in [x_j + 1, y_j - 1]} \binom{s_i + i - j}{s_i - (e_i - 1)} \binom{z_j + j - i}{e_i - 1}, & \text{if } j \in u \\ \displaystyle\binom{s_i + i - j}{s_i - e_i} \binom{y_j + j - i}{e_i}, & \text{if } j \in [1, p] \backslash u. \end{cases}$$

Then using the fact that

(87) det is a multilinear function of its columns we get

$$\text{(IV)} \qquad \det[H_{ij}] = \sum_{u \in J(p)} \det[H_{ij}^{(u)}]$$

For every $u \in J(p)$, $r \in X[u]$ and $e \in N(p, E)$, let

$$k_{ij}(r, e_i) = \begin{cases} \displaystyle\binom{s_i + i - j}{s_i - (e_i - 1)} \binom{r_j + j - i}{e_i - 1}, & \text{if } j \in u \\ \displaystyle\binom{s_i + i - j}{s_i - e_i} \binom{r_j + j - i}{e_i}, & \text{if } j \in [1, p] \backslash u. \end{cases}$$

Note that, by (26),
(88) if for any $i\in[1,p]$, $e_i<0$, then $k_{ij}(r,e_i)=0$, $\forall j\in[1,p]$.
Then again by (87) we get $\forall u\in J(p)$,

$$\det[H_{ij}^{(u)}]=\sum_{r\in X[u]}\det[k_{ij}(r,e_i)].$$

Hence by (III) and (IV) we get

$$H_E^{(2k)}(m^*,p^*,a^*)=\sum_{e\in N(p,E)}\sum_{u\in J(p)}\sum_{r\in X[u]}\det[k_{ij}(r,e_i)].$$

$$=\sum_{u\in J(p)}\sum_{r\in X[u]}\sum_{e\in N(p,E)}\det[k_{ij}(r,e_i)].$$

$$=\sum_{u\in J(p)}\sum_{r\in X[u]}\sum_{e\in N(p,E)}\det[k_{ij}(r,e_j)]. \qquad \text{(by (69))}$$

$$=\sum_{U\in[0,p]}\sum_{u\in J(p,U)}\sum_{r\in X[u]}\sum_{e\in N(p,E)}\det[k_{ij}(r,e_j)].$$

$$=\sum_{U\in[0,p]}\sum_{r\in X(U)}\sum_{e\in N(p,E)}\det[k_{ij}(r,e_j)]. \qquad \text{(by (73))}$$

$$=\sum_{U\in[0,p]}\sum_{a\in M(p,b,k,U)}\sum_{e\in N(p,E)}\det[k_{ij}(r[a],e_j)]. \qquad \text{(by (84))}$$

$$=\sum_{U\in[0,p]}\sum_{a\in M(p,b,k,U)}\sum_{e\in Z(p,E)}\det[k_{ij}(r[a],e_j)]. \qquad \text{(by (88))}$$

Now $\forall U\in[0,p]$, $u\in J(p,U)$ and $e\in Z(p,E)$ consider $d(u,e)\in Z(p)$ where

$$d(u,e)(j)=\begin{cases} e_j-1, & \text{if } j\in u\\ e_j, & \text{if } j\in[1,p]\backslash u\end{cases}$$

so that
$(*)$ $e\to d(u,e)$ is a bijection of $Z(p,E)$ onto $Z(p,E-U)$.
Then $\forall U\in[0,p]$, $u\in J(p,U)$, $a\in M[p,b,k,u]$ and $e\in Z(p,E)$ we see, by (51) that

$$\det[k_{ij}(r[a],e_j)]=H^{(2^*k)}(m,p,a,d(u,e)).$$

Also we then have

$$\sum_{e\in Z(p,E)}\det[k_{ij}(r[a],e_j)]=\sum_{e\in Z(p,E)}H^{(2^*k)}(m,p,a,d(u,e))$$

$$=\sum_{e\in Z(p,E-U)}H^{(2^*k)}(m,p,a,e). \qquad \text{(by (*))}$$

$$=H_{E-U}^{(2^*k)}(m,p,a)$$

$$=H_{E-U}^{(2k)}(m,p,a). \qquad \text{(by (70))}$$

Hence by the above

$$H_E^{(2k)}(m^*,p^*,a^*)=\sum_{U\in[0,p]}\sum_{a\in M(p,b,k,U)}H_{E-U}^{(2k)}(m,p,a)$$

Finally, if $p=1$ and $m(k)-a(k,p)=0$, we have

$$H_E^{(2k)}(m,p,a)=\binom{0}{-E}\binom{m(k')-a(k',1)}{E}$$
$$=\begin{cases}1, & \text{if } E=0\\ 0, & \text{if } E\in Z\backslash\{0\}.\end{cases}$$

Thus Theorem (3.3.1) is proved. □

(3.3.2) **Theorem.** *Let there be given* $p\in N^*$, $m\in N^*(2)$ *and* $a\in vec(m,p)$. *Then* $\forall k\in[1,2]$ *and* $\forall E\in Z$,

$$H_E^{(2k)}(m,p,a)=H_E^{(2^*k)}(m,p,a)$$
$$=|\mathrm{fr}(m,p,a;E)|$$
$=$ *number of frames in* $\mathrm{fr}(m,p,a)$ *each having exactly* E *antinodes.*

Proof. Follows from (3.2) and (3.3.1) in view of (65) and (70). □

Let p,m and a be as in the above theorem. Fix $D\in N$. For $E\in N$, consider any frame $W[a]$ in $\mathrm{fr}(m,p,a;E)$. Now D of the E antinodes in $W[a]$ can be chosen in $\binom{E}{D}$ different ways. For each such choice, mark the D chosen antinodes by red colour and call the frame obtained a D-marked frame. Let $A(E,D)$ denote the set of all D-marked frames obtained from various frames in $\mathrm{fr}(m,p,a;E)$. Then the set

$$B(D)=\bigcup_{E\in N}A(E,D),$$

where the union is disjoint, is the set of all D-marked frames obtainable from various frames in $\mathrm{fr}(m,p,a)$. Hence by the above theorem,

$$|B(D)|=\sum_{E\in N}|A(E,D)|=\sum_{E\in Z}\binom{E}{D}|\mathrm{fr}(m,p,a;E)|$$
$$=\sum_{E\in Z}\binom{E}{D}H_E^{(2k)}(m,p,a) \qquad (*)$$

(3.3.3) **Theorem.** *Let there be given* $p\in N^*$, $m\in N^*(2)$ *and* $a\in vec(m,p)$. *Then* $\forall k\in[1,2]$ *and* $\forall D\in Z$,

$F_D^{(2k)}(m,p,a)=|B(D)|=$ *number of D-marked frames obtainable from those in* $\mathrm{fr}(m,p,a)$.

Proof. Follows from $(*)$, since by definition, $\forall D\in Z$,

$$F_D^{(2k)}(m,p,a)=\sum_{E\in Z}\binom{E}{D}H_E^{(2k)}(m,p,a).$$

COROLLARY

(i) $F_D \geqslant 0$, *for all* $D \in N$, *where* $F_D = F_D(m, p, a)$.

(ii) *If for some* $E \in N^*$, $F_E = 0$ *then* $F_D = 0$ *for all* $D \geqslant E$.

(iii) *Let* $\bar{C}$ *be the maximum number of antinodes that a frame in* $\mathrm{fr}(m, p, a)$ *can have. Then*

$$F_D > 0 \Leftrightarrow D \in [0, \bar{C}].$$

(iv) *Let, for* $i \in [1, p]$,

$$r_i = m(1) - a(1, i), s_i = m(2) - a(2, i), \quad C_i = \min\{r_i, s_i\}$$

Then,

$$\bar{C} \leqslant \Sigma C_i.$$

Proof. (i) – (iii) are obvious. For (iv) note that in any frame $W \in \mathrm{fr}(m, p, a)$, the path $w(i)$ has at most r_i x-steps and s_i y-steps and hence at most C_i antinodes. Hence the frame W has atmost ΣC_i antinodes. □

Let the assumptions of (3.3.3) hold.

For all $E \in Z$, write

$$H_E^{(12)}(m, p, a) = H_E, \; H_E^{(22)}(m, p, a) = \hat{H}_E$$

and

$$F_D = F_D^{(12)}(m, p, a) = F_D^{(22)}(m, p, a)$$

and

$$s_i = m(2) - a(2, i), \; i \in [1, p] \text{ and } S = \sum_{i=1}^{p} s_i. \tag{89}$$

Then, by definition, $\forall D \in Z$,

$$F_D = \sum_{E \in Z} (-1)^{S-E} \binom{E}{D + E - S} H_E \tag{90}$$

and

$$F_D = \sum_{E \in Z} \binom{E}{D} \hat{H}_E. \tag{91}$$

We see from (2.1), (6) (i) and (89) above that in any frame the p paths have in all S right-end points in them. Hence, by (2), it follows that if in a frame j of these S right-end points are antinodes then the remaining $S - j$ are intermediate points. Now $S - E$ of these $S - j$ intermediate points can be chosen in $\binom{S-j}{S-E}$ different ways. For each such choice, mark the $S - E$ chosen intermediate points by blue colour and call the frame so obtained an E-labelled frame.

(3.3.4) **Theorem.** *Under the assumptions of* (3.3.3), $\forall E \in [0, S]$,

(i) $H_{S-E} = \sum_{i \in N} (-1)^i \binom{S-i}{E-i} F_i$

(ii) $H_E = \sum_{j \in N} \binom{S-j}{S-E} \hat{H}_j$

(iii) H_E = *number of E-labelled frames obtainable from those in* fr(m, p, a).

Proof. We need the following lemmas
(iv) ([1], Lemma 4.2′, page 96). For all $U, T, V \in Z$,

$$\binom{V}{U}\binom{U}{T} = \binom{V}{T}\binom{V-T}{U-T}$$

(v) Let there be given $n \in N$ and $x_i \in Q$ and $y_i \in Q$, $\forall i \in [0, n]$.
If $\forall j \in [0, n]$,

$$x_j = \sum_{k=0}^{n} (-1)^k \binom{k}{j} y_k,$$

then $\forall k \in [0, n]$,

$$y_k = \sum_{j=0}^{n} (-1)^j \binom{j}{k} x_j.$$

(vi) Let there be given $S, E, j \in N$ such that $j \leqslant S$ and $E \leqslant S$.
Then

$$\sum_{i \in N} (-1)^i \binom{S-i}{E} \binom{j}{i} = \binom{S-j}{S-E}.$$

Using (iv), (v) follows easily. For (vi) it is enough to show that for all $R \in [0, S]$,

$$\sum_{i \in N} (-1)^i \binom{S-i}{S-R} \binom{j}{i} = \binom{S-j}{R}.$$

Here, the generating function of left side is

$$\sum_{R \in N} \left[\sum_{i \in N} (-1)^i \binom{S-i}{S-R} \binom{j}{i} \right] x^R$$

$$= \sum_{i \in N} (-1)^i \binom{j}{i} \sum_{R \in N} \binom{S-i}{R-i} x^R \qquad \text{(by (27))}$$

$$= \sum_{i \in N} (-1)^i \binom{j}{i} \left[\binom{S-i}{0} x^i + \binom{S-i}{1} x^{i+1} + \ldots + \binom{S-i}{S-i} x^S \right]$$

$$= \sum_{i \in N} (-1)^i \binom{j}{i} x^i (1+x)^{s-i}$$

$$= (1+x)^S \sum_{i \in N} (-1)^i \binom{j}{i} \left(\frac{x}{1+x} \right)^i$$

$$= (1+x)^S \left(1 - \frac{x}{1+x} \right)^j$$

$$= (1+x)^{S-j}$$

$$= \sum_{R\in N} \binom{S-j}{R} x^R$$

= the generating function of right side.

Now $\forall D \in N$, by (27), (90) is

$$F_D = \sum_{E\in N} (-1)^{S-E} \binom{E}{S-D} H_E.$$

Hence applying (v), we obtain from

$$F_{S-D} = \sum_{E\in N} (-1)^{S-E} \binom{E}{D} H_E$$

that

$$H_E = \sum_{j\in N} (-1)^{S-j} \binom{j}{E} F_{S-j}.$$

Hence

$$H_{S-E} = \sum_{i\in N} (-1)^i \binom{S-i}{S-E} F_i$$

and so (i) follows.
Again,

$$H_E = \sum_{i\in N} (-1)^i \binom{S-i}{E} F_i$$

$$= \sum_{i\in N} (-1)^i \binom{S-i}{E} \sum_{j\in N} \binom{j}{i} \hat{H}_j \qquad \text{(by (91))}$$

$$= \sum_{j\in N} \left[\sum_{i\in N} (-1)^i \binom{S-i}{E} \binom{j}{i} \right] \hat{H}_j$$

$$= \sum_{j\in N} \binom{S-j}{S-E} \hat{H}_j. \qquad \text{(by (vi))}$$

Hence (ii) follows.
Finally, from (3.3.2) and (ii) above we obtain (iii).

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A note on two absolute summability methods

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Abstract. In this paper we have established a relation between $|R, p_n; \delta|_k$ and $|R, q_n; \delta|_k$ summability methods which generalizes the results of Bor [1] and Bosanquet [2].

Keywords. Absolute summability; summability methods.

1. Introduction

Let Σa_n be a given infinite series with the sequence of partial sums (s_n). By u_n we denote the nth $(C, 1)$ mean of the sequence (s_n). The series Σa_n is said to be summable $|C, 1; \delta|_k$, where $k \geqslant 1$ and $\delta \geqslant 0$, if (see [3])

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |u_n - u_{n-1}|^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \; (P_{-i} = p_{-i} = 0, \; i \geqslant 1). \tag{2}$$

The sequence-to-sequence transformation.

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v; \; (P_n \neq 0) \tag{3}$$

defines the sequence (t_n) of the (R, p_n) means of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [4]). We say that the series Σa_n is said to be summable $|R, p_n; \delta|_k$, where $k \geqslant 1$ and $\delta \geqslant 0$, if

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |t_n - t_{n-1}|^k < \infty. \tag{4}$$

In the special case when $p_n = 1$ for all values of n (resp. $\delta = 0$), $|R, p_n; \delta|_k$ summability is the same as $|C, 1; \delta|_k$ (resp. $|R, p_n|_k$) summability.

2. Regarding the relation between two absolute summability methods the following theorem due to Sunouchi [5] is given.

Theorem A. *Let (p_n) and (q_n) be positive sequences (where $Q_n = \Sigma_{v=0}^{n} q_v$). In order that every $|R, p_n|$ summable series should be $|R, q_n|$ summable it is sufficient that*

$$\frac{q_n P_n}{p_n Q_n} = O(1). \tag{5}$$

While reviewing this paper, Bosanquet [2] observed that the condition (5) is also necessary for conclusion and so completed Theorem A in necessary and sufficient form.

3. The aim of this paper is to generalize Bosanquet's result for $|R, p_n; \delta|_k$ and $|R, q_n; \delta|_k$ summability methods. Now, we shall prove the following theorem.

Theorem. *Let $k \geqslant 1$ and $\delta \geqslant 0$. In order that every $|R, p_n; \delta|_k$ summable series should be $|R, q_n; \delta|_k$ summable, the condition (5) is necessary. If we suppose that*

$$\sum_{n=v}^{\infty} \frac{n^{\delta k+k-1} q_n^k}{Q_n^k Q_{n-1}} = O\left(v^{\delta k+k-1} \frac{q_v^{k-1}}{Q_v^k} \right), \tag{6}$$

then the condition (5) is also sufficient.

Remark. If we take $k = 1$ and $\delta = 0$, in this theorem, then we get the observation of Bosanquet [2]. Also if we take only $\delta = 0$ in this theorem, then we get a result due to Bor [1].

4. We need the following lemma for the proof of our theorem.

Lemma. Let $k \geqslant 1$ and $A = (a_{nv})$ be an infinite matrix. In order that $A \in (l^k, l^k)$ it is necessary that $a_{nv} = O(1)$ for all $n, v \geqslant 0$.

Proof. This follows since $l^1 \subset l^k \subset l^\infty$ (see [1] for details).

5. *Proof of the theorem. Necessity.* We consider the series-to-series version of (3), i.e., for $n \geqslant 1$, let

$$b_n = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \tag{7}$$

$$c_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v. \tag{8}$$

A simple calculation shows that for $n \geqslant 1$

$$c_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{b_v}{p_v} (P_v Q_{v-1} - Q_v P_{v-1}) + \frac{q_n P_n b_n}{p_n Q_n}. \tag{9}$$

From this we can write down at once the matrix A transforms $\{n^{(\delta k+k-1)/k} b_n\}$ into $\{n^{(\delta k+k-1)/k} c_n\}$. Thus every $|R, p_n; \delta|_k$ summable series $|R, q_n; \delta|_k$ summable if and only if $A \in (l^k, l^k)$. By the lemma, it is necessary that the diagonal terms of A must be bounded, which gives that (5) must hold.

Sufficiency. Let $c_{n,1}$ denote the sum of the right of (9) and let $c_{n,2}$ denote the second term on the right of (9). Suppose the condition is satisfied, then, it is enough to show that, if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|b_n|^k < \infty, \tag{10}$$

we have

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|c_{n,i}|^k < \infty, \ (i=1,2). \tag{11}$$

For $i=2$ this is an immediate corollary of (5). Thus, to complete the proof of the sufficiency, it is enough to show that

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|c_{n,1}|^k < \infty.$$

we have

$$P_v Q_{v-1} - Q_v P_{v-1} = O(p_v Q_v), \text{ by (5)}.$$

Now, applying Hölder's inequality, with $k>1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\delta k+k-1}|c_{n,1}|^k &= O(1)\sum_{n=1}^{\infty} n^{\delta k+k-1}\frac{q_n^k}{(Q_n Q_{n-1})^k}\left\{\sum_{v=1}^{n-1}|b_v|Q_v\right\}^k \\
&= O(1)\sum_{n=1}^{\infty} n^{\delta k+k-1}\frac{q_n^k}{Q_n^k Q_{n-1}}\left\{\sum_{v=1}^{n-1}|b_v|^k q_v (Q_v/q_v)^k\right\}\left\{\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1} q_v\right\}^{k-1} \\
&= O(1)\sum_{n=1}^{\infty} n^{\delta k+k-1}\frac{q_n^k}{Q_n^k Q_{n-1}}\left\{\sum_{v=1}^{n-1}|b_v|^k q_v (Q_v/q_v)^k\right\} \\
&= O(1)\sum_{v=1}^{\infty}|b_v|^k q_v (Q_v/q_v)^k \sum_{n=v}^{\infty} n^{\delta k+k-1}\frac{q_n^k}{Q_n^k Q_{n-1}} \\
&= O(1)\sum_{v=1}^{\infty}|b_v|^k q_v (Q_v/q_v)^k v^{\delta k+k-1}\frac{q_v^{k-1}}{Q_v^k}, \text{ by (6)}
\end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|c_{n,1}|^k = O(1)\sum_{v=1}^{\infty} v^{\delta k+k-1}|b_v|^k < \infty, \text{ by (10)}.$$

This completes the proof of the theorem.

If we take $p_n = 1$ for all values of n, then $|R, p_n; \delta|_k$ summability reduces to $|C, 1; \delta|_k$ summability and the condition (5) reduces to

(i) $\qquad nq_n = O(Q_n)$.

Now, we obtain the following corollary from our theorem.

COROLLARY

Let $k \geqslant 1$ and $\delta \geqslant 0$. In order that every $|C, 1; \delta|_k$ summable series should be $|R, q_n; \delta|_k$ summable, (i) *is necessary. If the condition* (6) *is satisfied, then the condition* (i) *is also sufficient.*

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On deficiencies of differential polynomials

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Abstract. Bounds for deficiencies of differential polynomials have been given and some relations between Nevanlinna characteristic of differential polynomials in f and Nevanlinna characteristic function of f have been obtained.

Keywords. Meromorphic functions; differential polynomials; Nevanlinna theory and deficiencies.

1. Introduction

Let f be a non-constant meromorphic function in the complex plane. We use the usual notations $m(r,f)$, $n(r,f)$, $N(r,f)$, $T(r,f)$, $\bar{N}(r,f)$ and $S(r,f)$ etc of the Nevanlinna theory (see [2]). Let $P_0, P_1, \ldots, P_k$ be non-negative integers. Following Doeringer [1], we call

$$M[f] = f^{p_0}(f')^{p_1}\cdots(f^{(k)})^{p_k},$$

a monomial in f with $d_M = p_0 + p_1 + \ldots p_k$, its degree and $\Gamma_M = p_0 + 2p_1 + \ldots + (k+1)p_k$, its weight. Further let $M_1[f], M_2[f], \ldots, M_n[f]$ denote monomials in f and $a_1, a_2, \ldots, a_n$ meromorphic functions satisfying $T(r,a_j) = S(r,f)$, $1 \leqslant j \leqslant n$, then

$$Q[f] = \sum_{j=1}^{n} a_j M_j[f],$$

is called differential polynomial in f of degree $d_Q = \mathrm{Max}_{j=1}^{n} d_{M_j}$ and weight $\Gamma_Q = \mathrm{Max}_{j=1}^{n} \Gamma_{M_j}$ with coefficients a_j. For simplicity, we will denote d_{M_j} by d_j and Γ_{M_j} by Γ_j for all j. Throughout this paper we shall denote the quantity $[(d_1 + d_2 + \ldots d_n) - (n-1)d_Q]$ by A. Then clearly $A \leqslant d_Q$. If all the terms in $Q[f]$ have the same degree, then $Q[f]$ is called a homogeneous differential polynomial. As usual the Nevanlinna deficiency $\delta(a,f)$ and the Valiron deficiency $\Delta(a,f)$ are defined by

$$\delta(a,f) = \varliminf_{r\to\infty} \frac{m(r,a)}{T(r,f)} = 1 - \varlimsup_{r\to\infty} \frac{N(r,a)}{T(r,f)}$$

$$\Delta(a,f) = \varlimsup_{r\to\infty} \frac{m(r,a)}{T(r,f)} = 1 - \varliminf_{r\to\infty} \frac{N(r,a)}{T(r,f)}.$$

Also as usual

$$\Theta(a,f)=1-\overline{\lim_{r\to\infty}}\frac{\bar{N}(r,a)}{T(r,f)}.$$

In this paper we find bounds for deficiencies of differential polynomials and some relations between Nevanlinna characteristic of differential polynomials in f and Nevanlinna characteristic of f.

2. Statements of the results

We shall prove the following theorems:

Theorem 1. *Let f be a meromorphic function of finite order.*

(i) *If Q is a differential polynomial in f of degree d_Q and weight Γ_Q, then*

$$\overline{\lim_{r\to\infty}}\frac{T(r,Q)}{(r,f)}\leqslant\Gamma_Q-(\Gamma_Q-d_Q)\Theta(\infty,f).$$

(ii) *If Q is a differential polynomial of degree d_Q and if Q does not involve f, then*

$$A\sum_{b\neq\infty}\delta(b,f)\leqslant\delta(0,Q)\overline{\lim_{r\to\infty}}\frac{T(r,Q)}{T(r,f)}$$

and

$$A\sum_{b\neq\infty}\delta(b,f)\leqslant\Delta(0,Q)\underline{\lim_{r\to\infty}}\frac{T(r,Q)}{T(r,f)}.$$

The following theorem gives the relation between $\delta(0,Q)$ and $\delta(\alpha,f)$, $\alpha\neq\infty$:

Theorem 2. *Let f be a meromorphic function with $N(r,f)=S(r,f)$ (in particular for entire function f) and let $Q[f]$ be a differential polynomial of degree d_Q which does not contain the factor f, then*

$$\delta(0,Q)\geqslant\frac{A}{d_Q}\sum_{\alpha\neq\infty}\delta(\alpha,f).$$

Remark 2.1. If $Q=f'$ where f is an entire function then we obtain Theorem 4.6 of Hayman [2].

A consequence of Theorems 1 and 2 is the following corollary.

COROLLARY 1

If f is a meromorphic function of finite order with $N(r,f)+N(r,1/f)=S(r,f)$ and if Q is a differential polynomial in f which does not contain the factor f with

$$\underline{\lim_{r\to\infty}}\frac{N(r,1/Q)}{T(r,Q)}\geqslant 1-\frac{A}{d_Q},\ A\geqslant 0$$

then

$$\delta(0,Q)=\Delta(0,Q)=\frac{A}{d_Q}$$

and

$$T(r,Q)\sim d_Q T(r,f).$$

Remark 2.2. We note that the above corollary is not true if

$$\varliminf_{r\to\infty}\frac{N(r,1/Q)}{T(r,Q)}<1-\frac{A}{d_Q}$$

as shown by the following example:

Let $f(z)=e^z$ and $Q[f]=(f'''')^2+f'-(f''')^2$, then $Q[f]=e^z$. In this $\delta(0,Q)=1$ but $A/d_Q=1/2$.

Theorem 3. *Let f be a meromorphic function of finite order. Let $|\alpha_i|<\infty\,(i=1,2,\dots)$. If $Q[f]$ is a differential polynomial of degree d_Q which does not contain f then for $A>0$*

$$[1-\delta(O,Q)+\Delta(O,Q)]\sum_{i=1}^{\infty}\delta(\alpha_i,f)\leqslant\left[\frac{\Gamma_Q-(\Gamma_Q-d_Q)\Theta(\infty,f)}{A}\right]\Delta(O,Q).$$

The following theorem gives us the bounds for deficiencies of differential polynomials in terms of its corresponding degree and weight.

Theorem 4. *Let f be a meromorphic function of finite order with $\Sigma_{a\neq\infty}\delta(a,f)=2$. Then for any positive integer k, we have*

$$T(r,Q)\sim\Gamma_Q T(r,f),\ \delta(\infty,Q)=0,\ \delta(O,Q)\geqslant\frac{2A}{\Gamma_Q},$$

$$\sum_a\delta(a,Q)\leqslant\frac{\Gamma_Q+1}{\Gamma_Q}$$

where poles of $M_j[f]$, $1\leqslant j\leqslant n$ are different from zeros of $a_j(z)$ in $Q[f]=\Sigma_{j=1}^{n}a_j(z)M_j[f]$.

Remark 4.1. For $Q=f^{(k)}$, we get Theorem 1 of Xiao-Mao and Chong Ji [4].

Remark 4.2. Let $f(z)$ be a meromorphic function of finite order and let $\alpha_1\neq\alpha_2$ be two finite complex numbers. Let

$$\lim_{r\to\infty}\frac{N(r,\alpha_i)}{T(r,f)}=0\ (i=1,2)$$

then $T(r,Q)\sim\Gamma_Q T(r,f)$ where poles of $M_j[f]$ are not zeros of $a_j(z)$.

This follows since $\lim_{r\to\infty}(N(r,\alpha_i)/T(r,f))=0$ implies $\delta(\alpha_i)=1$ for $i=1,2$ so that $\Sigma_{i=1}^{\infty}\delta(\alpha_i)=2$ and $\delta(\infty,f)=0$.

3. Some lemmas

For the proofs of the above theorems we shall need the following lemmas.

Lemma 1 (see [1]). *Let f be a non-constant meromorphic function. If $Q[f]$ is a differential polynomial in f with arbitrary meromorphic coefficients q_j, $1 \leqslant j \leqslant n$ then*

$$m(r, Q[f]) \leqslant d_Q m(r,f) + \sum_{j=1}^{n} m(r, q_j) + S(r,f).$$

Lemma 2. *Let Q be a differential polynomial in f of degree d_Q having n terms and suppose that Q does not involve f. If Q is not a constant and $b_1, b_2, \ldots, b_q$ are distinct complex numbers (where q is any positive integer) then*

$$A \sum_{i=1}^{q} m(r, b_i, f) + N\left(r, \frac{1}{Q}\right) \leqslant T(r, Q) + S(r,f).$$

Proof of lemma 2. Let

$$F(z) = \sum_{i=1}^{q} \frac{1}{(f - b_i)^k}, \tag{1}$$

where k is a positive integer. Following Theorem 2.1 of Hayman [2] it is easy to see that

$$m(r, F) + O(1) \geqslant k \sum_{i=1}^{q} m(r, b_i, f). \tag{2}$$

Taking k as d_Q, we obtain by using Theorem 3.1 of Hayman ([2], p. 55) that

$$\begin{aligned} d_Q \sum_{i=1}^{q} m(r, b_i, f) &\leqslant m(r, F) + O(1) \\ &\leqslant m(r, 1/Q) + m(r, FQ) + S(r,f) \\ &\leqslant m(r, 1/Q) + \sum_{i=1}^{q} m\left(r, \frac{Q}{(f - b_i)^{d_Q}}\right) + S(r,f) \\ &\leqslant m(r, 1/Q) + \sum_{i=1}^{q} (d_Q - A) m(r, b_i, f) + S(r,f). \end{aligned}$$

This gives that

$$A \sum_{i=1}^{q} m(r, b_i, f) \leqslant m(r, 1/Q) + S(r,f),$$

which proves the desired result.

Lemma 3([3], Lemma 3). *Let $Q[f]$ be a non-constant differential polynomial. Let z_0 be a pole of f of order p and neither a zero nor a pole of coefficients of $Q[f]$. Then z_0 is a pole of $Q[f]$ of order atmost $pd_Q + (\Gamma_Q - d_Q)$.*

Lemma 4. *Let f be a meromorphic function of finite order. Then*

$$\{2-\delta(O,Q')\}\sum_{a\neq\infty}\delta(a,Q)+\frac{1}{\Gamma_Q}\delta(\infty,Q)\leqslant\frac{\Gamma_Q+1}{\Gamma_Q}.$$

Proof of lemma 4. We have, on replacing f by Q and $k=1$ in (1) and (2), that

$$F=\sum_{i=1}^{q}\frac{1}{Q-b_i}$$

and

$$m(r,F)+O(1)\geqslant\sum_{i=1}^{q}m(r,b_i,Q).$$

Therefore

$$\begin{aligned}N\left(r,\frac{1}{Q'}\right)+\sum_{i=1}^{q}m(r,b_i,Q)&\leqslant m(r,F)+N\left(r,\frac{1}{Q'}\right)+O(1)\\&\leqslant m(r,1/Q')+m(r,FQ')+N\left(r,\frac{1}{Q'}\right)+O(1)\\&\leqslant T(r,1/Q')+\sum_{i=l}^{q}m\left(r,\frac{Q'}{Q-b_i}\right)+O(1)\\&=T(r,Q')+S(r,Q)\\&=T(r,Q)-N(r,Q)+N(r,Q')+S(r,Q),\end{aligned}\tag{3}$$

which gives

$$\begin{aligned}&N\left(r,\frac{1}{Q'}\right)+\sum_{i=1}^{q}m(r,b_i,Q)+\frac{\Gamma_Q-1}{\Gamma_Q}N(r,Q)\\&\leqslant T(r,Q)+N(r,Q')-N(r,Q)+\frac{\Gamma_Q-1}{\Gamma_Q}N(r,Q)+S(r,Q).\end{aligned}$$

Since

$$N(r,Q')\leqslant\frac{\Gamma_Q+1}{\Gamma_Q}N(r,Q),$$

it follows that

$$N\left(r,\frac{1}{Q'}\right)+\sum_{i=1}^{q}m(r,b_i,Q)+\frac{\Gamma_Q-1}{\Gamma_Q}N(r,Q)\leqslant T(r,Q)+N(r,Q)+S(r,Q).$$

Hence we obtain

$$\varliminf_{r\to\infty}\frac{N\left(r,\frac{1}{Q'}\right)}{T(r,Q)}+\sum_{i=1}^{q}\delta(b_i,Q)+\frac{\Gamma_Q-1}{\Gamma_Q}(1-\delta(\infty,Q))\leqslant 2-\delta(\infty,Q).\tag{4}$$

Now from (3) we note that

$$\varliminf_{r\to\infty}\frac{T(r,Q')}{T(r,Q)}\geqslant\sum_{i=1}^{q}\delta(b_i,Q).$$

Therefore, we have

$$\varlimsup_{r\to\infty}\frac{N\left(r,\dfrac{1}{Q'}\right)}{T(r,Q)}\geqslant\sum_{i=1}^{q}\delta(b_i,Q)\left[\varlimsup_{r\to\infty}\frac{N\left(r,\dfrac{1}{Q'}\right)}{T(r,Q')}\right]$$

$$=[1-\delta(O,Q')]\sum_{i=1}^{q}\delta(b_i,Q).$$

Combining the above with (4), we have

$$[1-\delta(O,Q')]\sum_{i=1}^{q}\delta(b_i,Q)\leqslant 2-\delta(\infty,Q)-\sum_{i=1}^{q}\delta(b_i,Q)$$

$$-\frac{\Gamma_Q-1}{\Gamma_Q}(1-\delta(\infty,Q)),$$

which gives

$$[2-\delta(O,Q')]\sum_{i=1}^{q}\delta(b_i,Q)+\frac{1}{\Gamma_Q}\delta(\infty,Q)\leqslant\frac{\Gamma_Q+1}{\Gamma_Q}.$$

This proves Lemma 4.

Lemma 5. *Let f be a meromorphic function of finite order. If $\Sigma_{a\neq\infty}\delta(a,f)=2$, then $\bar{N}(r,f)\sim T(r,f)$ as $r\to\infty$.*

Proof. Replacing Q by f' and b_i by a_i in Lemma 2, we get

$$\sum_{i=1}^{q}m(r,a_i,f)\leqslant T(r,f')-N(r,1/f')+S(r,f).$$

Adding $\Sigma_{i=1}^{q}N(r,a_i,f)$ to both the sides, it follows that

$$qT(r,f)\leqslant T(r,f')+\sum_{i=1}^{q}N(r,a_i,f)-N(r,1/f')+S(r,f).$$

Dividing by $T(r,f)$ and taking limit inferior as $r\to\infty$, it follows that

$$q\leqslant\varliminf_{r\to\infty}\frac{T(r,f')}{T(r,f)}+\sum_{i=1}^{q}\varlimsup_{r\to\infty}\frac{N(r,a_i,f)}{T(r,f)}$$

$$=\varliminf_{r\to\infty}\frac{T(r,f')}{T(r,f)}+\sum_{i=1}^{q}(1-\delta(a_i,f)).$$

And so

$$\varliminf_{r\to\infty}\frac{T(r,f')}{T(r,f)}\geqslant\sum_{i=1}^{q}\delta(a_i,f).$$

Making $q\to\infty$, it follows that

$$\varliminf_{r\to\infty}\frac{T(r,f')}{T(r,f)}\geqslant 2. \tag{5}$$

Also

$$\varlimsup_{r\to\infty} \frac{T(r,f')}{T(r,f)} \leqslant 2.$$

Thus we have from (5),

$$\begin{aligned} T(r,f) &\sim \frac{1}{2} T(r,f') \text{ as } r\to\infty \\ &= \frac{1}{2}[m(r,f') + N(r,f')] \\ &\leqslant \frac{1}{2}\left[m\left(r,\frac{f'}{f}\right) + m(r,f) + N(r,f) + \bar{N}(r,f)\right] \\ &= \frac{1}{2}[T(r,f) + \bar{N}(r,f) + S(r,f)]. \end{aligned}$$

Thus

$$\frac{1}{2} T(r,f) \leqslant \frac{1}{2}\bar{N}(r,f) + S(r,f)$$

or equivalently

$$T(r,f) \leqslant \bar{N}(r,f) + S(r,f).$$

The lemma now follows since $\bar{N}(r,f) \leqslant T(r,f)$ for all r.

4. Proofs of the theorems

Proof of theorem 1. By Lemmas 1 and 3, we get

$$T(r,Q) \leqslant d_Q T(r,f) + (\Gamma_Q - d_Q)\bar{N}(r,f) + S(r,f). \tag{6}$$

Thus we obtain

$$\begin{aligned} \varlimsup_{r\to\infty} \frac{T(r,Q)}{T(r,f)} &\leqslant d_Q + (\Gamma_Q - d_Q) \varlimsup_{r\to\infty} \frac{\bar{N}(r,f)}{T(r,f)} \\ &= d_Q + (\Gamma_Q - d_Q)(1 - \Theta(\infty,f)) \\ &= \Gamma_Q - (\Gamma_Q - \delta_Q)\Theta(\infty,f). \end{aligned}$$

This proves part (i) of Theorem 1.

Next from Lemma 2 and noting that $S(r,Q) = S(r,f)$, we obtain

$$A \sum_{r\to\infty}^{q} \frac{m(r,b_i,f)}{T(r,f)} \leqslant \varliminf_{r\to\infty} \frac{m(r,1/Q)}{T(r,f)}.$$

Using the property that

$$\varliminf_{r\to\infty} f(r)g(r) \leqslant \varlimsup_{r\to\infty} f(r) \varliminf_{r\to\infty} g(r) \qquad \text{for } f(r) \geqslant 0$$

and $g(r) \geqslant 0$, it follows on making $q \to \infty$ that

$$A \sum_{b \neq \infty} \delta(b,f) \leqslant \delta(O,Q) \varlimsup_{r \to \infty} \frac{T(r,Q)}{T(r,f)}$$

and

$$A \sum_{b \neq \infty} \delta(b,f) \leqslant \varlimsup_{r \to \infty} \frac{m(r,1/Q)}{T(r,f)} \cdot \varliminf_{r \to \infty} \frac{T(r,Q)}{T(r,f)}$$

$$= \Delta(O,Q) \cdot \varliminf_{r \to \infty} \frac{T(r,Q)}{T(r,f)}.$$

This proves the required result.

Proof of theorem 2. Since by the hypothesis $N(r,f) = S(r,f)$, we have by (6) that

$$T(r,Q) \leqslant d_Q\, T(r,f) + S(r,f). \tag{7}$$

From Lemma 2 and using (7) we obtain

$$A \sum_{i=1}^{q} \varliminf_{r \to \infty} \frac{m(r,\alpha_i,f)}{d_Q\, T(r,f)} = \frac{A}{d_Q} \sum_{i=1}^{q} \delta(\alpha_i,f)$$

$$\leqslant \delta(O,Q).$$

Thus by making $q \to \infty$ we obtain

$$\delta(O,Q) \geqslant \frac{A}{d_Q} \sum_{i=1}^{\infty} \delta(\alpha_i,f).$$

Hence the result.

Proof of corollary 1. By Theorems 1 and 2, we obtain

$$\varlimsup_{r \to \infty} \frac{T(r,Q)}{T(r,f)} \leqslant \Gamma_Q - (\Gamma_Q - d_Q)\Theta(\infty,f),$$

$$\Delta(O,Q) \varliminf_{r \to \infty} \frac{T(r,Q)}{T(r,f)} \geqslant A \sum_{b \neq \infty} \delta(b,f)$$

and

$$\delta(O,Q) \geqslant \frac{A}{d_Q}.$$

Also by hypothesis, we have

$$\Delta(O,Q) \leqslant \frac{A}{d_Q}.$$

Thus we have

$$\delta(O,Q) = \Delta(O,Q) = \frac{A}{d_Q}.$$

Since $\Theta(\infty,f)=1$ and $\Sigma_{b\neq\infty}\delta(b,f)=1$, we have from the above

$$\overline{\lim_{r\to\infty}}\frac{T(r,Q)}{T(r,f)}\leqslant d_Q \text{ and } \underline{\lim_{r\to\infty}}\frac{T(r,Q)}{T(r,f)}\geqslant d_Q.$$

Thus we have

$$T(r,Q)\sim d_Q\,T(r,f).$$

This proves Corollary 1.

Proof of theorem 3. Let $B=\underline{\lim_{r\to\infty}}\dfrac{T(r,Q)}{T(r,f)}$ and $C=\overline{\lim_{r\to\infty}}\dfrac{T(r,Q)}{T(r,f)}$.

By Lemma 2 we have

$$Aq\,T(r,f)+N(r,1/Q)\leqslant T(r,Q)+A\sum_{i=1}^{q}N(r,\alpha_i,f)+S(r,f), \tag{8}$$

which gives

$$Aq+\underline{\lim_{r\to\infty}}\frac{N(r,1/Q)}{T(r,Q)}\,\underline{\lim_{r\to\infty}}\frac{T(r,Q)}{T(r,f)}\leqslant\underline{\lim_{r\to\infty}}\frac{T(r,Q)}{T(r,f)}+A\sum_{i=1}^{q}\overline{\lim_{r\to\infty}}\frac{N(r,\alpha_i,f)}{T(r,f)}.$$

Thus we have

$$Aq+[1-\Delta(O,Q)]B\leqslant B+A\sum_{i=1}^{q}[1-\delta(\alpha_i,f)] \tag{9}$$

which reduces to

$$A\sum_{i=1}^{q}\delta(\alpha_i,f)\leqslant B\Delta(O,Q). \tag{10}$$

Also from (8) we have

$$Aq+\overline{\lim_{r\to\infty}}\frac{N(r,1/Q)}{T(r,Q)}\,\underline{\lim_{r\to\infty}}\frac{T(r,Q)}{T(r,f)}\leqslant\overline{\lim_{r\to\infty}}\frac{T(r,Q)}{T(r,f)}+A\sum_{i=1}^{q}\overline{\lim_{r\to\infty}}\frac{N(r,\alpha_i,f)}{T(r,f)}.$$

So, we obtain

$$Aq+(1-\delta(O,Q))B\leqslant C+A\sum_{i=1}^{q}(1-\delta(\alpha_i,f)).$$

Therefore on rearranging we get

$$(1-\delta(O,Q))B\leqslant\left[C-A\sum_{i=1}^{q}\delta(\alpha_i,f)\right]. \tag{11}$$

From (10) and (11), we get

$$A(1-\delta(O,Q))\sum_{i=1}^{q}\delta(\alpha_i,f)\leqslant\left[C-A\sum_{i=1}^{q}\delta(\alpha_i,f)\right]\Delta(O,Q). \tag{12}$$

But by our supposition and Theorem 1, we have

$$C=\overline{\lim_{r\to\infty}}\frac{T(r,Q)}{T(r,f)}\leqslant \Gamma_Q-(\Gamma_Q-d_Q)\Theta(\infty,f). \tag{13}$$

From (12) and (13) we get

$$A(1-\delta(O,Q))\sum_{i=1}^{q}\delta(\alpha_i,f)\leqslant[\Gamma_Q-(\Gamma_Q-d_Q)\Theta(\infty,f)$$
$$-A\sum_{i=1}^{q}\delta(\alpha_1,f)]\Delta(O,Q).$$

Making $q\to\infty$ it follows that

$$A[1-\delta(O,Q)+\Delta(O,Q)]\sum_{i=1}^{\infty}\delta(\alpha_i,f)\leqslant[\Gamma_Q-(\Gamma_Q-d_Q)\Theta(\infty,f)]\Delta(O,Q).$$

This proves the desired result.

Proof of theorem 4. For the proof of Theorem 4, we put

$$B=\overline{\lim_{r\to\infty}}\frac{T(r,Q)}{T(r,f)},\ b=\underline{\lim}_{r\to\infty}\frac{T(r,Q)}{T(r,f)}.$$

By using (6) we have

$$\begin{aligned}B&\leqslant d_Q+(\Gamma_Q-d_Q)(1-\Theta(\infty,f))\\&=\Gamma_Q-(\Gamma_Q-d_Q)\Theta(\infty,f).\end{aligned} \tag{14}$$

On the other hand, by using Lemma 3 we have

$$\begin{aligned}b(1-\delta(\infty,Q))&=\underline{\lim}_{r\to\infty}\frac{T(r,Q)}{T(r,f)}\overline{\lim_{r\to\infty}}\frac{N(r,Q)}{T(r,Q)}\\&\geqslant\underline{\lim}_{r\to\infty}\frac{N(r,Q)}{T(r,f)}\\&\geqslant\underline{\lim}_{r\to\infty}\Gamma_Q\frac{\bar N(r,f)}{T(r,f)}\\&=\Gamma_Q-\Gamma_Q\bar\Delta(\infty,f)\end{aligned} \tag{15}$$

where

$$\bar\Delta(\infty,f)=1-\underline{\lim}_{r\to\infty}\frac{\bar N(r,f)}{T(r,f)}.$$

Since by hypothesis $\Sigma_{a\neq\infty}\delta(a,f)=2$, by using Lemma 5 it can be easily seen that $\Theta(\infty,f)=0$ and $\bar\Delta(\infty,f)=0$. Hence, it follows from (14) and (15) that $B\leqslant\Gamma_Q$ and $b(1-\delta(\infty,Q))\geqslant\Gamma_Q$. Thus

$$\Gamma_Q\leqslant\frac{\Gamma_Q}{1-\delta(\infty,Q)}\leqslant b\leqslant B\leqslant\Gamma_Q$$

i.e. $B=b=\Gamma_Q$.

Hence we have

$$T(r, Q[f]) \sim \Gamma_Q T(r, f) \text{ and } \delta(\infty, Q) = 0.$$

Therefore, by Lemma 4 we have

$$(2 - \delta(O, Q')) \sum_{a \neq \infty} \delta(a, Q) \leqslant \frac{\Gamma_Q + 1}{\Gamma_Q}$$

$$\text{i.e. } \sum_{a \neq \infty} \delta(a, Q) \leqslant \frac{\Gamma_Q + 1}{\Gamma_Q}.$$

Finally by Lemma 2, we have

$$[1 - \delta(O, Q)] + \frac{A}{\Gamma_Q} \sum_{i=1}^{q} \delta(a_i, f) \leqslant 1.$$

Consequently on making $q \to \infty$ we obtain

$$\delta(O, Q) \geqslant \frac{2A}{\Gamma_Q}.$$

Hence the result.

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A note on the identity operators of fractional calculus

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Abstract. The application of an identity operator for Saigo's fractional calculus operators is shown by evaluating the limit of an indeterminate form. Its special case yields the result which has been used as an infinitesimal generator in the semigroup theory. Also, an identity operator for the recently introduced multi-dimensional fractional operators (due to Srivastava and Raina [8]) is discussed.

Keywords. Fractional calculus operators; identity operator; Riemann–Liouville operators; Gauss hypergeometric function

1. Introduction and main result

In the theory of fractional calculus, a fractional integral operator $I_{ax}^{\alpha,\beta,\eta}$ due to Saigo [6] (see also [9] and [10]) is defined by

$$I_{ax}^{\alpha,\beta,\eta} f(x) = \frac{(x-a)^{-\alpha-\beta}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; \frac{x-t}{x-a}\right) f(t)\mathrm{d}t, \tag{1}$$

where $\alpha > 0$, β and η are real, $f(x)$ is any real continuous function defined on (a, ∞), with the order that

$$f(x) = O((x-a)^{\varepsilon}),\ x \to a,$$

where

$$\varepsilon > \max\{0, \beta - \eta\} - 1.$$

Here F is the Gauss hypergeometric function defined by ([7])

$$F(a, b; c; z) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r r!} z^r, \tag{2}$$

in $|z| < 1$ (and its analytic continuation into $|\arg(1-z)| < \pi$), with $(a)_r$ denoting the usual Pochhammer symbol $(a)_r = \Gamma(a+r)/\Gamma(a)$.

As stated in [6], there exists the relationship

$$I_{ax}^{0,0,\eta} f(x) = f(x) \tag{3}$$

which shows that $I_{ax}^{0,0,\eta}$ serves as an identity operator. A logistically permissive proof of (3) can be given by following Ross [4, pp. 13–15], and we skip here these details.

We examine the following limit of an indeterminate form involving the operator (1):

$$\Delta = \lim_{\alpha\to 0^+} \frac{(x-a)^{2\alpha} I_{ax}^{\alpha,\alpha,\eta} f(x) - f(x)}{\alpha}$$

$$= \lim_{\alpha\to 0^+} \frac{1}{\alpha}\left[\frac{1}{\Gamma(a)}\int_a^x (x-t)^{\alpha-1} F\left(2\alpha, -\eta; \alpha; \frac{x-t}{x-a}\right) f(t)\mathrm{d}t - f(x)\right], \tag{4}$$

where $f(x)$ is continuous and differentiable on (a,x) and $\alpha > 0$.

In view of (3), (4) takes the indeterminate form (0/0) when $\alpha = 0$. Integration by parts leads (4) to

$$\Delta = \lim_{\alpha\to 0^+} \frac{1}{\alpha}\left[\frac{1}{\Gamma(1+\alpha)}\left\{(x-t)^{\alpha} F\left(2\alpha, -\eta; 1+\alpha; \frac{x-t}{x-a}\right) f(t)\right\}_x^a\right.$$
$$\left. + \frac{1}{\Gamma(1+\alpha)}\int_a^x (x-t)^{\alpha} F\left(2\alpha, -\eta; 1+\alpha; \frac{x-t}{x-a}\right) f'(t)\mathrm{d}t - f(x)\right]$$
$$= \lim_{\alpha\to 0^+} \frac{1}{\alpha}\left[\frac{(x-a)^{\alpha}\Gamma(1-\alpha+\eta) f(a)}{\Gamma(1-\alpha)\Gamma(1+\alpha+\eta)}\right.$$
$$\left. + \frac{1}{\Gamma(1+\alpha)}\int_a^x (x-t)^{\alpha} F\left(2\alpha, -\eta; 1+\alpha; \frac{x-t}{x-a}\right) f'(t)\mathrm{d}t - f(x)\right], \tag{5}$$

by virtue of the summation formula [7, p. 28, (1.7.6)].

Clearly (5) is again of the indeterminate form (0/0) when $\alpha = 0$, and by L' Hôspital's rule, we find that

$$\Delta = \lim_{\alpha\to 0^+} \frac{1}{\alpha\Gamma'(1+\alpha) + \Gamma(1+\alpha)}\left[\frac{(x-a)^{\alpha}\Gamma(1+\alpha)\Gamma(1-\alpha+\eta) f(a)}{\Gamma(1-\alpha)\Gamma(1+\alpha+\eta)}\right.$$
$$\times \{\psi(1+\alpha) - \psi(1-\alpha+\eta) + \psi(1-\alpha) - \psi(1+\alpha+\eta) + \log(x-a)\}$$
$$\left. + \frac{\partial p}{\partial \alpha} - \Gamma'(1+\alpha) f(x)\right], \tag{6}$$

where

$$P = \int_a^x (x-t)^{\alpha} F\left(2\alpha, -\eta; 1+\alpha; \frac{x-t}{x-a}\right) f'(t)\,\mathrm{d}t, \tag{7}$$

and

$\psi(z) = \Gamma'(z)/\Gamma(z)$, is the psi-function.

By differentiating under the integral sign with Leibnitz rule, we get

$$\frac{\partial P}{\partial \alpha} = \int_a^x (x-t)^{\alpha} \log(x-t) F\left(2\alpha, -\eta; 1+\alpha; \frac{x-t}{x-a}\right) f'(t)\,\mathrm{d}t$$
$$+ \int_a^x (x-t)^{\alpha} f'(t)\left[\sum_{r\geqslant 0} \frac{(2\alpha)_r(-\eta)_r}{(1+\alpha)_r r!}\{2\psi(2\alpha+r) + \psi(1+\alpha)\right.$$
$$\left. - \psi(1+\alpha+r) - 2\psi(2\alpha)\}\left(\frac{x-t}{x-a}\right)^r\right]\mathrm{d}t. \tag{8}$$

Evidently when $\alpha \to 0^+$, the last term in (8) above vanishes, and we get

$$\lim_{\alpha\to 0^+} \frac{\partial P}{\partial \alpha} = \int_a^x \log(x-t) f'(t)\,\mathrm{d}t. \tag{9}$$

Finally, in view of (4), (6) and (9) give

$$\begin{aligned}\lim_{\alpha\to 0^+} \frac{1}{\alpha}\left[\frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} F\left(2\alpha, -\eta; \alpha, \frac{x-t}{x-a}\right) f(t)\,\mathrm{d}t - f(x)\right] \\ = f(a)\{2\psi(1) - 2\psi(1+\eta) + \log(x-a)\} - f(x)\psi(1) \\ + \int_a^x \log(x-t) f'(t)\,\mathrm{d}t.\end{aligned} \tag{10}$$

A special case of (10) when $a = \eta = 0$, is the result obtained recently by Ross and Sachdeva [5, p. 205]. This special case of eq. (10) giving the limit of the difference quotient was described as an infinitesimal generator in semi-group theory in [1], and obtained in an L_p setting by means of functional analysis.

2. Multidimensional operator

In a recent paper, Srivastava and Raina [8] have introduced a multidimensional extension of the familiar differential operator ${}_cD_z^q$ (see [4] for its definition). This extension is defined by ([8, p. 357, (1.6)]; see also [2] and [3])

$$\begin{aligned}{}_{\tilde{c}}D_{x_1,\ldots,x_n}^{q_1,\ldots,q_n} f(x_1,\ldots,x_n) = {}_{c_1}D_{x_1}^{q_1} \cdots {}_{c_n}D_{x_n}^{q_n} f(x_1,\ldots,x_n) \\ = \frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots \partial x_n^{m_n}} \int_{c_1}^{x_1} \cdots \int_{c_n}^{x_n} f(t_1,\ldots,t_n) \prod_{i=1}^{n} \left\{ \frac{(x_i-t_i)^{m_i-q_i-1}}{\Gamma(m_i-q_i)} \mathrm{d}t_i \right\},\end{aligned} \tag{11}$$

where $m_i > \mathrm{Re}(q_i)$, $m_i \in N_0 \equiv \{0,1,2,\ldots\}$, for $i = 1,\ldots,n$; and for convenience $\tilde{c} = (c_1,\ldots,c_n)$. It must be pointed out here that different subscripts on D in (11) require different classes of functions to have a convergent integral. The multidimensional extensions of the Riemann-Liouville and Weyl operators of the fractional calculus are defined in [8].

If the function $f(x_1,\ldots,x_n)$ is continuous on each of the interval (c_i, x_i) for $i = 1,\ldots,n$; then

$${}_{\tilde{c}}D_{x_1,\ldots,x_n}^{m_1,\ldots,m_n} f(x_1,\ldots,x_n) = \frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots \partial x_n^{m_n}} f(x_1,\ldots,x_n), \tag{12}$$

where

$$m_i \in N_0 (i = 1,\ldots,n).$$

To prove the assertion (12), we write (11) as

$$\begin{aligned}{}_{\tilde{c}}D_{x_1,\ldots,x_n}^{q_1,\ldots,q_n} f(x_1,\ldots,x_n) \equiv I_1 + I_2 \\ = \frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots \partial x_n^{m_n}} \int_{c_1}^{x_1} \cdots \int_{c_n}^{x_n} [f(t_1,\ldots,t_n) - f(x_1,\ldots,x_n)]\end{aligned}$$

$$\times \prod_{i=1}^{n}\left\{\frac{(x_i-t_i)^{m_i-q_i-1}}{\Gamma(m_i-q_i)}\mathrm{d}t_i\right\}+\frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}}\int_{c_1}^{x_1}\cdots\int_{c_n}^{x_n}f(x_1,\ldots,x_n)$$
$$\times \prod_{i=1}^{n}\left\{\frac{(x_i-t_i)^{m_i-q_i-1}}{\Gamma(m_i-q_i)}\mathrm{d}t_i\right\}. \tag{13}$$

Introducing points $\delta_i > 0$ $(i=1,\ldots,n)$ in the interval of integrations then I_1 can be expressed as the sum of two n-tuple integrals

$$\begin{aligned}I_1 &= \frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}}\int_{c_1}^{x_1-\delta_1}\cdots\int_{c_n}^{x_n-\partial_n}\prod_{i=1}^{n}\left\{\frac{(x_i-t_i)^{m_i-q_i-1}}{\Gamma(m_i-q_i)}\mathrm{d}t_i\right\}\\ &\quad\times[f(t_1,\ldots,t_n)-f(x_1,\ldots,x_n)]\\ &\quad+\frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}}\int_{x_1-\delta_1}^{x_1}\cdots\int_{x_n-\delta_n}^{x_n}\prod_{i=1}^{n}\left\{\frac{(x_i-t_i)^{m_i-q_i-1}}{\Gamma(m_i-q_i)}\mathrm{d}t_i\right\}\\ &\quad\times[f(t_1,\ldots,t_n)-f(x_1,\ldots,x_n)]\\ &=\Delta_1+\Delta_2 \text{ (suppose).}\end{aligned} \tag{14}$$

For the multiple integral Δ_2, let

$$M=\max_{\substack{x_i-\delta_i\leqslant t_i\leqslant x_i\\(i=1,\ldots,n)}}|f(t_1,\ldots,t_n)-f(x_1,\ldots,x_n)|,$$

then $|f(t_1,\ldots,t_n)-f(x_1,\ldots,x_n)|\leqslant M$, where M depends on x_i and δ_i, and for fixed x_i, $M\to 0$ as $\delta_i\to 0$, for all $i=1,\ldots,n$ (because f is continuous). We now have

$$\begin{aligned}|\Delta_2| &\leqslant M\cdot\frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}}\int_{x_1-\delta_1}^{x_1}\cdots\int_{x_n-\delta_n}^{x_n}\prod_{i=1}^{n}\left\{\frac{(x_i-t_i)^{m_i-q_i-1}}{\Gamma(m_i-q_i)}\mathrm{d}t_i\right\}\\ &= M\cdot\frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}}\prod_{i=1}^{n}\left\{\frac{\delta_i^{m_i-q_i}}{\Gamma(1+m_i-q_i)}\right\}\to 0,\end{aligned} \tag{15}$$

as $\delta_i\to 0$ $(i=1,\ldots,n)$.

Similarly for the first multiple integral Δ_1 in (14), let

$$m=\max_{\substack{c_i\leqslant t_i\leqslant x_i-\delta_i\\(i=1,\ldots,n)}}|f(t_1,\ldots,t_n)-f(x_1,\ldots,x_n)|,$$

then

$$|\Delta_1|\leqslant m\cdot\frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}}\prod_{i=1}^{n}\left\{\frac{\delta_i^{m_i-q_i}-(x_i-c_i)^{m_i-q_i}}{\Gamma(1+m_i-q_i)}\right\}\to 0, \tag{16}$$

as $q_i\to m_i$, for fixed $\delta_i (i=1,\ldots,n)$.

The value of I_2 in (3) is easily given by

$$I_2=\frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}}\left[f(x_1,\ldots,x_n)\prod_{i=1}^{n}\left\{\frac{(x_i-c_i)^{m_i-q_i}}{\Gamma(1+m_i-q_i)}\right\}\right]. \tag{17}$$

Adding

$$-\frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}}f(x_1,\ldots,x_n)$$

to both the sides of (13) and using (14) and (17), we find that

$$\begin{aligned}&\left|{}_{\bar{c}}D^{q_1,\ldots,q_n}_{x_1,\ldots,x_n} f(x_1,\ldots,x_n) - \frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}} f(x_1,\ldots,x_n)\right| \\ &\quad \leqslant |\Delta_1| + |\Delta_2| + \left|\frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}} f(x_1,\ldots,x_n)\left[\prod_{i=1}^{n}\left\{\frac{(x_i-c_i)^{m_i-q_i}}{\Gamma(1+m_i-q_i)}\right\}-1\right]\right|.\end{aligned} \tag{18}$$

If ε is an arbitrary positive number, then choosing each $\delta_i (i=1,\ldots,n)$ such that $|\Delta_2| < \varepsilon$, we have (in view of (16) and (18))

$$\limsup_{\substack{q_i\to m_i\\(1\leqslant i\leqslant n)}} \left|{}_{\bar{c}}D^{q_1,\ldots,q_n}_{x_1,\ldots,x_n} f(x_1,\ldots,x_n) - \frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}} f(x_1,\ldots,x_n)\right| \leqslant \varepsilon. \tag{19}$$

By letting $\varepsilon \to 0$ (since ε is arbitrary), the assertion (12) follows.

For $m_i = 0$ $(i=1,\ldots,n)$, (12) gives the identity operator

$${}_{\bar{c}}D^{0,\ldots,0}_{x_1,\ldots,x_n} f(x_1,\ldots,x_n) = f(x_1,\ldots,x_n). \tag{20}$$

for the multidimensional fractional operators.

A direct proof of (12) can be given by first integrating by parts the multiple integral which defines (11), and then resorting to the process of limits when each $q_i \to m_i$ $(i=1,\ldots,n)$. We leave to the interested reader further details concerning this alternative proof of the assertion (12).

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An RSA based public-key cryptosystem for secure communication

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Abstract. A new cryptosystem that uses modulo arithmetic operations is proposed. It is based on Rivest–Shamir–Adleman's public key cryptosystem. A feature of the proposed system is that the encryption and decryption procedures are computationally less intensive, and hence the system is amenable for high data bit rate communications.

Keywords. Cryptanalysis; high data bit rate communication; encryption; decryption.

1. Introduction

In his paper, Shannon [6] indicated the need to develop new cryptosystems as they would improve the level of security of the existing cryptosystems. In this article, we propose a cryptosystem [8] that uses Rivest–Shamir–Adleman's system (RSA) to bootstrap into the new encryption scheme designed here.

It is known that the much celebrated RSA public-key cryptosystem [4] cannot be used in high data bit rate communication systems as the encryption and decryption procedures of RSA are computationally involved. Even the less known ElGamal's public-key cryptosystem [1] is computationally prohibitive for high data bit rate situations. Also, the size of the ciphertext of this system is double the size of the message.

The encryption scheme designed here is computationally less intensive and hence can be used in high data bit rate communication systems.

The proposed system is explained in § 2 and § 3 discusses some issues in system design. An example to illustrate the system is given in § 4 and analysis of the attacks is considered in § 5. Finally, an algorithm to compute the multiplicative inverse of an integer is explained in Appendix.

2. Proposed system

In this system, the sender chooses the set of integers p_i and g_j, $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 6$, that satisfy the conditions given in (1), uses the encryption procedure of RSA with the public-key, given in the public directory or obtained from the receiver to encrypt the chosen set of integers and obtains the corresponding ciphertext. This ciphertext is then transmitted to the intended receiver. The receiver applies the decryption scheme of RSA with the secret key known only to him, to the received ciphertext and obtains

the set of integers p_i and g_j. After transferring the chosen set of integers, the sender then uses (3) that involves certain modulo arithmetic operations on the chosen set of integers and the message blocks m_i, $i \geqslant 1$, obtains the corresponding ciphertext c_{9+i}, and transmits the ciphertext c_{9+i} to the receiver. By performing the reverse modulo arithmetic operations specified in (4), the receiver recovers the message blocks. Formally, the system is as follows.

Public Key

N : integer that has large prime factors P and Q,
e : multiplicative inverse of d modulo $\phi(N)$, Euler's totient function of N

Secret Key

d : integer less than $\phi(N)$ and is relatively prime to $\phi(N)$,
P and Q : prime factors of N,
$\phi(N)$: Euler's totient function of N and is equal to $(P-1)*(Q-1)$

Encryption

Choose the set of integers p_i, $1 \leqslant i \leqslant 3$, (not necessarily be prime) and g_j, $1 \leqslant j \leqslant 6$, such that they satisfy the following requirements.

$$p_i < p_j \quad \text{whenever} \quad i < j,$$
$$gcd(p_i, g_i) = 1 \quad \text{for all} \quad i. \tag{1}$$

Use RSA encryption procedure with the public-key e and N, to encrypt the chosen set of integers and obtain the corresponding ciphertext c_i, i.e. compute

$$c_i = p_i^e \bmod N \quad \text{for all } i$$
$$c_{3+j} = g_j^e \bmod N \quad \text{for all } j \tag{2}$$

where p_i and g_j are assumed to be less than N. Otherwise, each integer should be split into a set of small integers which are less than N.

Transmit the ciphertext to the intended receiver.

Partition the message into blocks so that the numeric value m_i of each block is less than p_1. Now encrypt each message block m_i, using the relation

$$c_{9+i} = (((m_i + g_4)g_1 \bmod p_1 + g_5)g_2 \bmod p_2 + g_6)\bar{g}_3 \bmod p_3, \tag{3}$$

obtain the ciphertext c_{9+i}, and transmit c_{9+i}.

Decryption

Receive $c_i, i < 10$, and compute, using RSA decryption procedure with secret key d,

$$c_i^d \bmod N = p_i^{ed} \bmod N = p_i \quad \text{for all} \quad i = 1, 2, 3,$$
$$c_{3+j}^d \bmod N = g_j^{ed} \bmod N = g_j \quad \text{for all} \quad j = 1, \ldots, 6$$

to obtain the set of integers p_i and g_j.

Use the following relation

$$m_i = (((((c_{9+i} g_3^{-1}(p_3) - g_6) \bmod p_3) g_2^{-1}(p_2) - g_5) \bmod p_2) g_1^{-1}(p_1) - g_4) \bmod p_1 \tag{4}$$

to recover the message blocks m_i from the ciphertext blocks c_{9+i}. $g_i^{-1}(p_i)$ is the mutliplicative inverse of g_i with respect to p_i.

3. Issues in system design

For the choice of prime numbers P and Q, and the secret key d in RSA, refer the book by Salomaa [5]. d should be large and relatively prime to $\phi(N)$. If d is small, direct search with reasonable amount of computation will reveal the key. To check whether the chosen integer d is relatively prime to $\phi(N)$ or not, generate $\{r_i\}$ as explained in the appendix by taking $a = \phi(N)$ and $b = d$. If $\{r_n\} = 1$, then d is relatively prime to $\phi(N)$, else not.

Since d and $\phi(N)$ are relatively prime, the multiplicative inverse e of d modulo $\phi(N)$ exists. e can be computed using Euclid's algorithm given in the appendix.

The sender can choose the integers p_i and g_j randomly. These integers should be from the set of large integers. Otherwise, the frequency of the plaintext characters may be preserved in the ciphertext. Also, direct search may reveal the key (integers p_i and g_j) of the message encryption scheme.

Coming to the computational aspect of encryption and decryption procedures, generation of ciphertext c_i, $i \geqslant 10$, in the encryption requires three multiplications, three additions, and three divisions. To compute c_i, $1 \leqslant i \leqslant 9$, exponentiation by repeated squaring and multiplication [4] may be adopted. This procedure requires atmost $2\log_2 e$ multiplications, $2\log_2 e$ divisions. For more efficient procedures, see [3].

Exponentiation by repeated squaring and multiplication method can also be employed in decryption to obtain p_i and g_j. Note that, since d is known only to the receiver, a cryptanalyst cannot use this procedure. To recover the message from c_i, $i \geqslant 10$, the receiver should first compute $g_i^{-1}(p_i)$ for each $i = 1, 2, 3$, say, by Euclid's algorithm which will terminate in atmost $\log_f p_i$ iterations [7], where $f = (1 + \sqrt{5})/2$ is the golden ratio. Each iteration requires one division with remainder operation. Before the start of communication, the sender may compute $g_i^{-1}(p_i)$ and transmit $g_i^{-1}(p_i)$ instead of g_i. The receiver can then recover the message in just three multiplications, three subtractions and three divisions.

If need arises, the system may be modified to amortize the number of arithmetic operations, by dropping g_4, g_5, g_6, without sacrificing the level of security of the system.

The proposed system is explained in the following example.

4. Example

Let $m_1 = 301$, $m_2 = 250$ and $m_3 = 276$ be three blocks of message to be transmitted.

Public key

$$N = 391$$
$$e = 3$$

Secret Key

$$d = 235$$
$$P = 17 \text{ and } Q = 23$$
$$\phi(N) = 352$$

Encryption

Choose $p_1 = 317$, $p_2 = 323$, $p_3 = 371$ and $g_1 = 41$, $g_2 = 47$, $g_3 = 11$, $g_4 = 10$, $g_5 = 13$, $g_6 = 8$. Note that $p_i < p_j$ whenever $i < j$ and $\gcd(p_i, g_i) = 1$ for all $i = 1, 2, 3$. Use RSA with public-key $e = 3$ and $N = 391$ to encrypt $p_1 = 317$ and obtain the ciphertext

$$c_1 = p_1^e \bmod N = 317^3 \bmod 391 = 243.$$

Similarly, encrypt $p_2, p_3, g_1, \ldots, g_6$ and obtain the ciphertext $c_2 = 323$, $c_3 = 211$, $c_4 = 205$, $c_5 = 208$, $c_6 = 158$, $c_7 = 218$, $c_8 = 242$, $c_9 = 121$.

Transmit the ciphertext c_i, $1 \leqslant i \leqslant 9$, to the intended receiver. This is one time transmission and the integers p_i and g_j become the key for the message encryption scheme.

Encrypt the first message block $m_1 = 301$ and obtain the ciphertext

$$\begin{aligned} c_{10} &= (((m_1 + g_4)g_1 \bmod p_1 + g_5)g_2 \bmod p_2 + g_6)g_3 \bmod p_3 \\ &= (((301 + 10)41 \bmod 317 + 13)47 \bmod 323 + 8)11 \bmod 371 \\ &= 138. \end{aligned}$$

Repeat this for the remaining message blocks $m_2 = 250$, $m_3 = 276$ and obtain the ciphertext $c_{11} = 134$, $c_{12} = 301$.

Decryption

Use RSA with secret key $d = 235$ to decrypt $c_1 = 243$ and obtain

$$p_1 = c_1^d \bmod N = 243^{235} \bmod 391 = 317.$$

Similarly, decrypt $c_2, c_3, \ldots, c_9$ and obtain $p_2, p_3, g_1, \ldots, g_6$.

Receive the ciphertext c_{10} and recover the message m_1 by computing

$$\begin{aligned} m_1 &= (((((c_{10} g_3^{-1}(P_3) - g_6) \bmod p_3) g_2^{-1}(p_2) \\ &\quad - g_5) \bmod p_2) g_1^{-1}(p_1) - g_4) \bmod p_1 \\ &= (((((138 * 135 - 8) \bmod 371)55 - 13) \bmod 323)116 - 10) \bmod 317 \\ &= 301, \end{aligned}$$

where $g_1^{-1}(p_1) = 41^{-1}(317) = 116$, $g_2^{-1}(p_2) = 47^{-1}(323) = 55$, $g_3^{-1}(p_3) = 11^{-1}(371) = 135$. Similarly, compute the messages $m_2 = 250$ and $m_3 = 276$ from the ciphertexts c_{11} and c_{12} respectively.

5. Cryptanalysis of the proposed system

The proposed system is analyzed against possible cryptanalytic approaches and found that the system is secure.

It is easy to see that computing p_i and g_j from c_i, $1 \leqslant i \leqslant 9$, is equivalent to breaking RSA. Hence it is assumed that p_i and g_j cannot be derived from c_i, $1 \leqslant i \leqslant 9$. Therefore, security of the proposed system is equivalent to security of the proposed encryption scheme.

Ciphertext only attack

Upon receiving the ciphertext, an intruder may be able to partition the ciphertext into blocks that reveal the size of p_3 and hence p_3.

Also, an intruder, though almost impossible, may find maximum number of plaintext characters corresponding to blocks of ciphertext; knowing the size of p_1 and hence p_1. To prevent this leakage of information on the size of p_1, partition the message so that each block m_i is much smaller than p_1.

But p_2 and g_j, $1 \leqslant j \leqslant 6$, can be obtained only by exhaustive search, which is computationally not feasible.

Known and chosen plaintext attacks

To analyze the system under known plaintext attack and chosen plaintext attack, write c_i, $i \geqslant 10$, in terms of the message and the key, i.e. write

$$\begin{aligned} c_{9+i} = {} & m_i g_1 g_2 g_3 + g_1 g_2 g_3 g_4 + g_2 g_3 g_5 + g_3 g_6 - k_3(m_i) p_3 \\ & - k_2(m_i) p_2 g_3 - k_1(m_i) p_1 g_2 g_3, \quad i \geqslant 1, \end{aligned}$$

where $k_j(m_i)$ are the quotients in the mod operation with p_j. This may conveniently be written as

$$m_i a + b(m_i) = c_{9+i} \quad i \geqslant 1,$$

where

$$\begin{aligned} a &= g_1 g_2 g_3, \\ b(m_i) &= g_1 g_2 g_3 g_4 + g_2 g_3 g_5 + g_3 g_6 - k_3(m_i) p_3 - k_2(m_i) p_2 g_3 \\ & \quad - k_1(m_i) p_1 g_2 g_3. \end{aligned}$$

Let J be a subset of positive integers. In the known plaintext attack, cryptanalyst is assumed to have knowledge of (m_i, c_{9+i}) for all $i \in J$, but not a and $b(m_i)$ even when p_3 is assumed to be known. He then wants to solve for m_i, $i \notin J$. Since there are $|J| + 1$, $|J|$ is the cardinality of J, equations in $|J| + 3$ variables, the system is under determined and has large number of solutions. Hence the system is secure against the known plaintext attack.

Similarly, the system can be proved to be secure against the chosen plaintext attack. The scheme also seems to be secure against the threat of solving for an equivalent key.

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Appendix (Multiplicative inverse computation)

Let $a > b > 0$ be two integers. The following procedure (Euclid's algorithm) computes the greatest common divisor of a and b, $\gcd(a, b)$. If $\gcd(a, b) = 1$, a and b are relatively prime, then the procedure also computes the multiplicative inverse of b modulo a. By division property

$$
\begin{aligned}
a &= b*q_1 + r_1, & 0 < r_1 < b,\\
b &= r_1*q_2 + r_2, & 0 < r_2 < r_1\\
r_1 &= r_2*q_3 + r_3, & 0 < r_3 < r_2\\
&\vdots\\
r_{n-2} &= r_{n-1}*q_n + r_n, & 0 < r_n < r_{n-1}\\
r_{n-1} &= r_n*q_{n+1}
\end{aligned}
$$

where, $r_n \neq 0$ (but $r_{n+1} = 0$). The greatest common divisor of a and b is then r_n. Now to compute the multiplicative inverse of b modulo a, define

$$
\begin{aligned}
y_1 &= 0 - 1*q_1,\\
y_2 &= 1 - y_1*q_2,\\
y_3 &= y_1 - y_2*q_3,\\
y_4 &= y_2 - y_3*q_4,\\
&\vdots\\
y_n &= y_{n-2} - y_{n-1}*q_n,\\
y_{n+1} &= y_{n-1} - y_n*q_{n+1}.
\end{aligned}
$$

If $r_n = 1$, the multiplicative inverse of b modulo a is then the positive residue of y_n modulo a. If $r_n > 1$, then $\gcd(a, b) = r_n > 1$ implying a and b are not relatively prime to each other and hence the multiplicative inverse of b modulo a does not exist. $\{r_i\}$, $\{y_i\}$ can be computed as in the following table [2]:

	a	0
	b	1
q_1	r_1	y_1
q_2	r_2	y_2
	$\vdots$	
q_n	r_n	y_n
q_{n+1}	r_{n+1}	y_{n+1}

As an example, let a be 352 and b be 235. Then the computation of the multiplicative inverse of b modulo a is

	352	0
	235	1
1	117	−1
2	1	3
117	0	−352

Since $r_2 = 1{,}235^{-1}(352) = 3$. As a check, consider

$$235 * 3 \bmod 352 = 705 \bmod 352$$
$$= 1.$$

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A note to the paper "An efficient algorithm for linear programming" of V Ch Venkaiah

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Abstract. In Venkaiah [1] an algorithm for solving linear optimization problems based on the idea of the projective algorithm of Karmarkar, is proposed. The essential simplification in the new algorithm is the use of a fixed projection operator. In this way the algorithm requires only $O(n^2)$ operations to obtain a sufficient exact solution. In this note it is shown that in some special cases the algorithm of Venkaiah yields a feasible solution that is far from the optimal one.

1. The algorithm of Venkaiah

Let us consider the linear optimization problem, which is denoted by (P2) in [1]:

$$\begin{aligned} C^t X &\to \min \\ AX &= b \\ X &\geqslant 0 \end{aligned} \tag{P2}$$

with C, $X \in R^n$, $b \in R^m$ and A as a $[m, n]$-matrix. Let

$$A^+ = A^t(AA^t)^{-1}$$

be the generalized inverse to A.
The algorithm proposed by Venkaiah [1] is the following:

Algorithm A2

Step 1: Compute an initial feasible solution $X^0 > 0$.
Step 2: Compute the projection operator $P = I - A^+ A$.
Step 3: Compute $C_p = PC$ and $Y = C_p/\|C_p\|$.
Step 4: Set $k = 0$.
Step 5: Compute

$$\lambda = \min \{X_i^h/Y_i\colon\ Y_i > 0\}.$$

The problem (P2) is unbounded if all $Y_i \leqslant 0$ and at least one $Y_i < 0$. If all $Y_i = 0$ then X^k is a solution.

Step 6: Compute $X^{k+1} = X^k - \varepsilon\lambda Y$ where $0 < \varepsilon < 1$.
Step 7: If

$$\|X^{k+1} - X^k\| < 2^{-L},$$

where L is defined by Karmarkar, then stop.
Step 8: Set $k = k+1$, $D_k = \operatorname{diag}(d_{ii})$ with

$$d_{ii} = \frac{X_i^k}{\sqrt{P_{ii}}}.$$

Step 9: Compute

$$Y = \frac{PD_kC_p}{\|PD_kC_p\|},$$

go to step 5.

2. The convergence of the algorithm

In [1] in theorem 1 it is formulated that this algorithm A2 converges to an (optimal) solution of problem (P2). The proof of this theorem contains the following steps:

(i) $X^k \geqslant 0$ for every k.
(ii) $AX^k = b$ for every k.
(iii) $C^tX^1 = C^tX^0 - \varepsilon\lambda\|C_p\| < C^tX^0$ and $C^tX^{k+1} = C^tX^k - \varepsilon\lambda(C_p^tD_kC_p/\|PD_kC_p\| < C^tX^k$ for every $k = 1, 2, \ldots$

Since in [1] (P2) is assumed to have a bounded solution it follows that C^tX^k is bounded below. Hence C^tX^k converges:

$$C^tX^k \to \tilde{C}, \tilde{C} \in R^1.$$

In [1] without any proof it is assumed that this value $\tilde{C}$ is the optimal value C^* of objective function in (P2).

But $\tilde{C}$ must not be equal to the optimal value C^*. The following example shows that the case $\tilde{C} > C^*$ is possible.

3. Counter example

Consider the following problem:

$$\begin{aligned} -x_1 - x_2 &\to \min \\ 2x_1 - x_2 + x_3 &= 4 \\ -x_1 + 2x_2 + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geqslant 0. \end{aligned}$$

The optimal solution is $X^* = (4; 4; 0; 0)^t$ with the optimal value $C^* = -8$.
Now we use the proposed algorithm A2.

Step 1: $X^0 = (0{\cdot}1;\ 2{\cdot}0;\ 5{\cdot}8;\ 0{\cdot}1)^t$
Step 2: We compute

$$\mathbf{P} = \frac{1}{10}\begin{pmatrix} 3 & 2 & -4 & -1 \\ 2 & 3 & -1 & -4 \\ -4 & -1 & 7 & -2 \\ -1 & -4 & -2 & 7 \end{pmatrix}$$

Step 3: $C_p = PC = \frac{1}{10}(-5;\ -5;\ 5;\ 5)^t = \frac{1}{2}(-1;\ -1;\ 1;\ 1)^t$
Step 4: $k = 0$
Step 5: $\lambda = \min\{5{\cdot}8/0{\cdot}5;\ 0{\cdot}1/0{\cdot}5\} = 0{\cdot}2$
Step 6: With $\varepsilon = 0{\cdot}5$ we obtain

$$X^1 = (0{\cdot}1;\ 2{\cdot}0;\ 5{\cdot}8;\ 0{\cdot}1)^t - 0{\cdot}5 * 0{\cdot}2 * 0{\cdot}5 * (-1;\ -1;\ 1;\ 1)^t$$
$$= (0{\cdot}15;\ 2{\cdot}05;\ 5{\cdot}75;\ 0{\cdot}05)^t$$

Step 7: —
Step 8: $D_1 = \mathrm{diag}(0{\cdot}27386;\ 3{\cdot}74277;\ 6{\cdot}87256;\ 0{\cdot}05976)$
Step 9: $Y = (-0{\cdot}53832;\ -0{\cdot}28356;\ 0{\cdot}79308;\ 0{\cdot}02879)^t$
Step 5: $\lambda = 1{\cdot}7363$
Step 6: $X^2 = (0{\cdot}61734;\ 2{\cdot}29617;\ 5{\cdot}06149;\ 0{\cdot}025)^t$
Step 7: —
Step 8: $D_2 = \mathrm{diag}(1{\cdot}12711;\ 4{\cdot}19221;\ 6{\cdot}04964;\ 0{\cdot}02988)$
Step 9: $Y = (-0{\cdot}54434;\ -0{\cdot}31758;\ 0{\cdot}77109;\ 0{\cdot}09087)^t$
Step 5: $\lambda = 0{\cdot}275$
Step 6: $X^3 = (0{\cdot}69226;\ 2{\cdot}33988;\ 4{\cdot}95537;\ 0{\cdot}0125)^t$
⋮
Step 6: $X^4 = (0{\cdot}72633;\ 2{\cdot}36004;\ 4{\cdot}90738;\ 0{\cdot}00625)^t$
⋮
Step 6: $X^5 = (0{\cdot}74270;\ 2{\cdot}36979;\ 4{\cdot}88438;\ 0{\cdot}003125)^t$
⋮
Step 6: $X^6 = (0{\cdot}75074;\ 2{\cdot}37459;\ 4{\cdot}87310;\ 0{\cdot}00156)^t$
⋮
⋮
Step 6: $\tilde{X} = (0{\cdot}75856;\ 2{\cdot}37927;\ 4{\cdot}86214;\ 0{\cdot}00002)^t$.

We see that the algorithm converges, but the solution is far from the optimal. With another value $0 < \varepsilon < 1$ we have the same effect. If we start the algorithm with another point

$$X^0 = (2;\ 2;\ 2;\ 2)^t$$

after some iterations the algorithm yields a solution

$$\tilde{X} = (3{\cdot}99988;\ 3{\cdot}99988;\ 0{\cdot}00012;\ 0{\cdot}00012)^t$$

which is near to the optimal solution.

4. Conclusion

The algorithm A2 converges to a feasible solution. The quality of this solution depends on the starting point. Other conditions as described in [1] are necessary to guarantee

the convergence to an optimal solution. Then the algorithm will, however, require more than $O(n^2)$ operations.

Reference

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Stochastic dilation of minimal quantum dynamical semigroup

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Abstract. A necessary and sufficient condition is formulated for minimal quantum dynamical semigroups to be conservative. The paper also provides a Markovian dilation of the minimal semigroups, as a contractive solution of an associated quantum stochastic differential equation in Boson-Fock space, which is isometric if and only if the minimal semigroup is conservative. Using the reflection principle of Brownian motion a necessary and sufficient condition for the contractive solution to be co-isometric is also obtained.

Keywords. Quantum dynamical semigroup; Markovian cocycle; quantum stochastic differential equation.

1. Introduction

Feller [8] proved the existence of a unique minimal semigroup $P_t, t \geqslant 0$ on l_1 associated with the Fokker-Planck equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{ik}(t) = \sum_j p_{ij}(t)\Omega_{jk}, t \geqslant 0, p_{ik}(0) = \delta_{ik} \tag{1}$$

subject to the Markov condition:

$$\Omega_{jk} \geqslant 0 \text{ for } j \neq k \quad \text{and} \quad \sum_{k \neq j} \Omega_{jk} = -\Omega_{jj} < \infty. \tag{2}$$

Exploiting the special nature of l_1, Kato [14] constructed the minimal semigroup in the framework of semigroup theory. It was also shown in [8, 14] that the minimal semigroup is conservative i.e. $\|P_t y\|_1 = \|y\|_1$, for all $y \in l_1^+$ if and only if

$$B_\lambda \equiv \{x \in l_\infty^+, \sum_k \Omega_{jk} x_k = \lambda x_j\} = \{0\}, \quad \text{for some } \lambda > 0. \tag{3}$$

In this paper we consider the quantum mechanical Fokker-Planck equation in $\mathscr{T}$, the Banach space of trace class operators in $\mathscr{H}_0$:

$$\rho(0) = \rho, \quad \rho(t)' = Y\rho(t) + \rho(t)\, Y^* + \sum_{k \in S} Z_k \rho(t) Z_k^* \tag{4}$$

subject to

$$Y + Y^* + \sum_{k \in S} Z_k^* Z_k = 0, \tag{5}$$

where Y, $Z_k, k \in S \subset \mathbb{Z}_+$ are densely defined operators in $\mathscr{H}_0$ and $\rho \in \mathscr{T}_h$, the real Banach space of self-adjoint elements in $\mathscr{T}$. Davies [4], following essentially Kato's method, constructed the minimal dynamical semigroup $\sigma_t^{\min}(t \geqslant 0)$ in $\mathscr{T}_h$ as a solution to (4)–(5). In this context, we formulate a condition similar to (3) as the necessary and sufficient one for the preservation of trace under the action of $\sigma_t^{\min}$. We also provide a Markovian dilation of $\sigma^{\min}$ in the sense of Accardi [1,2] as a contractive solution of an associated Hudson–Parthasarathy equation [11,12,19]. The solution is isometric if and only if $\sigma^{\min}$ is trace preserving. Finally using Journé's reflection principle [13,17] we also obtain a necessary and sufficient condition for the contractive solution to be co-isometric. Some results on the related dilation problem may be found in Chebotarev [3] and Fagnola [6,7]. The method employed here is different from that in [3,7].

The paper is organized as follows: In §2 we describe the framework of quantum stochastic calculus and a class of contractive cocycles satisfying quantum stochastic differential equation (qsde) with bounded coefficients and also recall [11, 17, 20] the necessary and sufficient condition for the solution to be isometric, co-isometric or unitary. Section 3 is devoted to exactly the same questions as in §2, this time with unbounded coefficients subject to some conditions. Many of the results in this section are quoted without proof since they are published elsewhere [19]. In §4 we consider the problem mentioned at the beginning.

2. Contractive bar-cocycles

All the Hilbert spaces that appear here are assumed to be complex and separable with inner product $\langle \cdot, \cdot \rangle$ linear in the second variable. For any Hilbert space H, we denote by $\Gamma(H)$ the symmetric Fock space over H and $B(H)$ the C^* algebra of all bounded linear operators in H. For any $u \in H$, we denote by $e(u)$ the exponential vector in $\Gamma(H)$ associated with u. The family $\{e(u): u \in \mathscr{M}\}$ is total for any dense linear manifold $\mathscr{M}$ in H and linearly independent in $\Gamma(H)$.

We fix two Hilbert spaces $\mathscr{H}_0$ and k and write

$$\mathscr{H} = \mathscr{H}_0 \otimes \Gamma(L^2(\mathbb{R}'_+, k)).$$

It is clear that for any pair of linear manifolds $\mathscr{D}$ and $\mathscr{M}$ dense in $\mathscr{H}_0$ and $L^2(\mathbb{R}_+, k)$ respectively, the algebraic tensor product $\mathscr{D} \otimes \varepsilon(\mathscr{M})$ is dense in $\mathscr{H}$, where $\varepsilon(\mathscr{M})$ is the linear manifold generated by the vectors $e(u): u \in \mathscr{M}$. We also denote the vacuum conditional expectation on $\mathscr{H}_0$ by $\mathbb{E}_0$.

For the basic notions in boson stochastic calculus such as *adapted, regular, bounded, contractive, isometric, co-isometric* and *unitary* process, we refer to [11,21]. The notion of Markovian cocycle was first introduced in [1]. However in this paper we follow the definition introduced in [13] and call it *bar-cocycle* to avoid confusion.

We fix an orthonormal basis $\{e_i: i \in S\}$ in k and set $E^i_j = |e_j\rangle\langle e_i| : i, j \in S$. With respect to this basis we define the basic quantum stochastic processes $\{\Lambda^i_j : i, j \in \bar{S} := S \cup \{0\}\}$ as in [18, 23]. Then quantum Ito's formula [11] can be expressed as:

$$\mathrm{d}\Lambda^i_j \mathrm{d}\Lambda^k_\ell = \hat{\delta}^i_\ell \mathrm{d}\Lambda^k_j \tag{6}$$

for all $i, j, k, l \in \bar{S}$ where

$$\hat{\delta}^i_\ell = \begin{cases} 0 & \text{if } \ell = 0 \text{ or } i = 0 \\ \delta^i_\ell & \text{otherwise.} \end{cases}$$

We denote by $u^j(s) = \langle e_j, u(s) \rangle$, $u_j(s) = \overline{u^j(s)}$ for $j \in S$ and $u_0(s) = u^0(s) = 1$. Choose $\mathcal{M} \equiv \{u \in H : u^j(\cdot) = 0$ for all but finitely many $j \in S\}$ and set $N(u) = \{j; u^j(\cdot) \not\equiv 0\}$. So $\# N(u) < \infty$ for $u \in \mathcal{M}$.

We also denote by $\mathscr{L}_R$ the class of elements $L \equiv (L^i_j \in \mathscr{B}(\mathscr{H}_0), i, j \in \bar{S})$ such that for each $j \in \bar{S}$ there exists a non-negative constant c_j (depending on L) satisfying

$$\sum_{i \in \bar{S}} \| L^i_j f \|^2 \leqslant c_j^2 \| f \|^2 \tag{7}$$

for all $f \in \mathscr{H}_0$. For any $L \in \mathscr{L}_R$ define the family of bounded linear operators $\{\mathscr{L}^i_j, i, j \in \bar{S}\}$ on $\mathscr{H}_0$ by

$$\mathscr{L}^i_j = L^i_j + (L^j_i)^* + \sum_{k \in S} (L^k_i)^* L^k_j$$

where the necessary convergence follows from (7).

Fix $L \in \mathscr{L}_R$. Then there exists a unique adapted process $X \equiv \{X(t), t \geqslant 0\}$ satisfying the following qsde:

$$\mathrm{d}X = \sum_{i,j \in \bar{S}} L^i_j \mathrm{d}\Lambda^j_i(t) X(t), \; X(0) = I \tag{8}$$

on $\mathscr{H}_0 \otimes \varepsilon(\mathcal{M})$. Moreover X is isometric whenever $L \in \mathscr{I}_R$, where $\mathscr{I}_R \equiv \{L \in \mathscr{L}_R, \mathscr{L}^i_j = 0$, for all $i, j \in \bar{S}\}$. For a complete account of these facts the reader is referred to [11, 16, 17, 18, 20, 21].

Observe that for all $i, j \in \bar{S}$, $\mathscr{L}^i_j = (\mathscr{L}^j_i)^*$ and $\mathscr{L}_{S'} \equiv ((\mathscr{L}^i_j))_{i,j \in S'}$ is a self adjoint operator on the Hilbert space $\mathscr{H}_0 \otimes l_2(S')$ for any finite subset S' of S. We set

$$\mathscr{L}^-_R \equiv \{L, \mathscr{L}_{S'} \leqslant 0, \quad \text{for all } S' \subset S, \# S' < \infty\}.$$

Hence $\mathscr{I}_R \subset \mathscr{L}^-_R \subset \mathscr{L}_R$.

The following proposition gives a necessary and sufficient condition for X to be contractive.

PROPOSITION 2.1.

Fix $L \in \mathscr{L}_R$. Consider the family $X \equiv \{X(t), 0 \leqslant t < \infty\}$ of operators satisfying (8). *The following statements are valid:*

(i) *X has a contractive extension if and only if $L \in \mathscr{L}^-_R$;*
(ii) *X has an isometric extension if and only if $L \in \mathscr{I}_R$.*

Proof. By (6) and (8) we have

$$\begin{aligned} &\langle X(t) f e(u), X(t) g e(v) \rangle - \langle f e(u), g e(v) \rangle \\ &= \int_0^t \left\langle X(\tau) f e(u), \sum_{i,j \in \bar{S}} u_i(\tau) v^j(\tau) \mathscr{L}^i_j X(\tau) g e(v) \right\rangle \mathrm{d}\tau, 0 \leqslant t \end{aligned} \tag{9}$$

for all $f, g \in \mathscr{H}_0, u, v \in \mathscr{M}$. For finitely many vectors $f_\alpha \in \mathscr{H}_0$, ${}^\alpha u \in \mathscr{M}$ let $\psi := \Sigma_\alpha f_\alpha e({}^\alpha u) \cdot \| e({}^\alpha u) \|^{-1}$. It is convenient to introduce the Hilbert space $H = \oplus_\alpha H_\alpha$ with $H_\alpha = \oplus_{j \in N({}^\alpha u)} \mathscr{H}$, the vector in $H: \Psi(t) = \oplus_\alpha \psi_{f_\alpha, {}^\alpha u}(t)$ with $\psi_{f,u}(t) = \oplus_{j \in N(u)} u^j(t) X(t) f e(u) \cdot \| e(u) \|^{-1}$, and the bounded operator $\mathbb{L}$ in $H: \mathbb{L}^\alpha_\beta = \mathscr{L}$ for all α, β. Then from (9) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \| X(t)\psi \|^2 = \langle \Psi(t), \mathbb{L}\Psi(t) \rangle. \tag{10}$$

Also observe that $\mathscr{L}$ is negative semi-definite if and only if $\mathbb{L}$ is negative semi-definite. Hence from (10) it is clear that the map $t \to \| X(t)\psi \|, t \geqslant 0$ is decreasing whenever $L \in \mathscr{Z}^-_R$. This completes the proof of the sufficiency part of (i). Conversely, let X be contractive so that $(\mathrm{d}/\mathrm{d}t) \| X(t)\psi \|^2_{|t=0} \leqslant 0$. Fix any finite set of vectors $g_\alpha \in \mathscr{H}_0$, $\alpha \in S'$, where $S' \subset \bar{S}, \# S' < \infty$. Taking continuous functions ${}^\alpha u \in \mathscr{M}$ so that ${}^\alpha u^j(0) = \delta^\alpha_j$ and

$$f_\alpha = \begin{cases} g_\alpha, & \text{if } \alpha \neq 0, \\ g_0 - \sum\limits_{\beta \neq 0} g_\beta, & \text{if } 0 \in S', \alpha = 0 \\ - \sum\limits_{\beta \neq 0} g_\beta, & \text{if } 0 \notin S', \alpha = 0 \end{cases}$$

in (10) we have

$$\sum_{\alpha, \beta \in S'} \langle g_\alpha, \mathscr{L}^\alpha_\beta g_\beta \rangle \leqslant 0.$$

Hence $L \in \mathscr{Z}^-_R$. This completes the proof of (i). The proof of (ii) is very similar to that of (i). ■

For any $L \equiv (L^i_j : i, j \in \bar{S})$ with L^i_j densely defined closed operators in $\mathscr{H}_0$ we define $\tilde{L} \equiv \{\tilde{L}^i_j : i, j \in \bar{S}\}$ by

$$\tilde{L}^i_j = (L^j_i)^*, \quad i, j \in \bar{S}.$$

and set

$$\tilde{\mathscr{Z}}_R \equiv \{L, \tilde{L} \in \mathscr{Z}_R\}, \quad \tilde{\mathscr{Z}}^-_R = \{L, \tilde{L} \in \mathscr{Z}^-_R\} \text{ and } \tilde{\mathscr{I}}_R = \{L, \tilde{L} \in \mathscr{I}_R\}.$$

As a consequence of Proposition 2.1 and 'time reversal principle' [17], we have the following theorem, which we state without proof. The proof can be found in [17, 20].

Theorem 2.2. *Suppose $Z \in \mathscr{Z}_R \cap \tilde{\mathscr{Z}}_R$. Then there exists a unique regular $(\mathscr{H}_0, \mathscr{M})$-adapted process $V = \{V(t), 0 \leqslant t < \infty\}$ satisfying*

$$dV(t) = \sum_{i, j \in \bar{S}} V(t) Z^i_j d\Lambda^j_i(t), \quad V(0) = I \tag{11}$$

on $\mathscr{H}_0 \otimes \varepsilon(\mathscr{M})$. Moreover the following hold:

(i) *The following statements are equivalent: (a) V has a contractive extension; (b) $Z \in \mathscr{Z}^-_R$; (c) $Z \in \tilde{\mathscr{Z}}^-_R$.*

In such a case V is a strongly continuous bar-cocycle.

(ii) *V has an isometric extension if and only if $Z \in \mathscr{I}_R$;*

(iii) *V has a co-isometric extension if and only if* $Z\in\tilde{\mathscr{I}}_R$;
(iv) *V has a unitary extension if and only if* $Z\in\mathscr{I}_R\cap\tilde{\mathscr{I}}_R$.

3. A class of qsde with unbounded coefficients

In this section we recall some results from [19] which will enable us to deal with more general quantum evolutions satisfying (11) on $\mathscr{D}\otimes\varepsilon(\mathscr{M})$, where $\mathscr{D}$ is a common dense domain of the family $Z\equiv\{Z^i_j, i,j\in\bar{S}\}$ of operators in the initial Hilbert space $\mathscr{H}_0$.

We denote by $\mathscr{Z}^-(\mathscr{D})$ the class of elements $Z\equiv\{Z^i_j, i,j\in\bar{S}\}$ such that Z^0_0 is the generator of a strongly continuous contractive semigroup with $\mathscr{D}$ as a core and assume furthermore that

$$\text{(a) } \mathscr{D}\subseteq\mathscr{D}(Z^i_j);\quad (i,j\in\bar{S}); \tag{12}$$

(b) there exists a sequence $Z(n)\in\mathscr{Z}^-_R\cap\tilde{\mathscr{Z}}^-_R, n\geqslant 1$ so that for all $f\in\mathscr{D}$, $i,j\in\bar{S}$

$$s=\lim_{n\to\infty} Z^i_j(n)f=Z^i_j f. \tag{13}$$

Let $Z\in\mathscr{Z}^-(\mathscr{D})$. From Lemma 3.1 in [19] we observe that for each $f\in\mathscr{D}, j\in\bar{S}$ there exists a constant $c_j(f)\geqslant 0$ such that

$$\sup_{n\geqslant 1}\sum_{i\in\bar{S}}\|Z^i_j(n)f\|^2\leqslant c_j(f) \tag{14}$$

and

$$\sum_{i\in\bar{S}}\|Z^i_j f\|^2\leqslant c_j(f). \tag{15}$$

For any $X\in\mathscr{B}(\mathscr{H}_0)$ we define the bilinear forms $\mathscr{L}^i_j(X)(i,j\in\bar{S})$ on $\mathscr{D}$ by

$$\langle f,\mathscr{L}^i_j(X)g\rangle=\langle f,XZ^i_j g\rangle+\langle Z^j_i f,Xg\rangle+\sum_{k\in S}\langle Z^k_i f,XZ^k_j g\rangle$$

where the necessary convergence follows from (15) and Cauchy–Schwarz inequality. We set for $\lambda>0$

$$\beta_\lambda\equiv\{X\geqslant 0: X\in\mathscr{B}(\mathscr{H}_0);\quad \mathscr{L}^0_0(X)=\lambda X\},$$

and denote by $\mathscr{I}$ the class of elements $Z\in\mathscr{Z}^-(\mathscr{D})$ such that

$$\mathscr{L}^i_j(I)=0\quad\text{for all } i,j\in\bar{S}.$$

Fix a sequence $Z(n)\in\mathscr{Z}^-_R\cap\tilde{\mathscr{Z}}^-_R$ satisfying (12) and (13). We denote by $V^{(n)}\equiv\{V^{(n)}(t): t\geqslant 0\}$ the unique regular $(\mathscr{H}_0,\mathscr{M})$ adapted contractive process satisfying (11) with $Z(n)$ as its coefficients. We state the following propositions without proof, referring to [19, 20] for the proofs.

PROPOSITION 3.1.

Let $Z\in\mathscr{Z}^-(\mathscr{D})$ and $V^{(n)}$ be as above. Then
(i) $w-\lim_{n\to\infty}V^{(n)}(t)=V(t)$ *exists for all* $t\geqslant 0$,

(ii) $V = \{V(t): t \geqslant 0\}$ *is the unique strongly continuous contractive barcocycle satisfying* (11) *on* $\mathscr{D} \otimes \varepsilon(\mathscr{M})$;
(iii) V *is isometric only if* $Z \in \mathscr{I}$;
(iv) *if* $Z \in \mathscr{I}$ *and* $\beta_\lambda \equiv \{0\}$ *for some* $\lambda > 0$ *then* V *is isometric.*

Remark 3.2. Suppose for each $n \geqslant 1$, $V^{(n)}$ is a regular contractive $(\mathscr{H}_0, \mathscr{M})$-adapted process satisfying (11) on $\mathscr{D} \underline{\otimes} \varepsilon(\mathscr{M})$ where $Z(n)$ are densely defined operators on $\mathscr{D}$. Then Proposition 3.1 holds as well for the associated sequence $V^{(n)}$ provided (12)–(14) are valid for Z. We omit the proof since it follows by the method employed for the proof of Proposition 3.3 in [19].

Let $\tilde{Z} \in \mathscr{Z}^-(\tilde{\mathscr{D}})$, for some a dense linear manifold $\tilde{\mathscr{D}}$ in $\mathscr{H}_0$. We denote by $\tilde{\mathscr{I}}$ and $\tilde{\beta}_\lambda$ the classes $\mathscr{I}$ and β_λ respectively, with Z replaced by $\tilde{Z}$.

COROLLARY 3.3.

Consider the contractive cocycle V *defined as in Proposition 3.1. Let in addition* $\tilde{Z} \in \mathscr{Z}^-(\tilde{\mathscr{D}})$. *Then the following hold:*
(i) *if* V *is co-isometric then* $Z \in \tilde{\mathscr{I}}$;
(ii) if $Z \in \tilde{\mathscr{I}}$ *and* $\tilde{\beta}_\lambda \equiv \{0\}$ *for some* $\lambda > 0$ *then* V *is co-isometric.*

4. Minimal quantum dynamical semigroup and its dilation

We consider the quantum mechanical Fokker-Planck equation written formally as

$$\rho(0) = \rho, \quad \rho(t)' = Y\rho(t) + \rho(t)\,Y^* + \sum_{k \in S} Z_k \rho(t) Z_k^* \tag{16}$$

subject to

$$Y + Y^* + \sum_{k \in S} Z_k^* Z_k \leqslant 0 \tag{17}$$

for $\rho \in \mathscr{T}_h$, where $Y, Z_k, k \in S$ are densely defined operators in $\mathscr{H}_0$ and $\mathscr{T}_h$ is the real Banach space of all self-adjoint trace class operators in $\mathscr{H}_0$. When Y is a bounded operator, (17) implies that $\{Z_k, k \in S\}$ is a family of bounded operators and the series $\Sigma_{k \in S} Z_k^* Z_k$ converges in strong operator topology. In such a case, for each ρ (16) admits a unique $\mathscr{T}_h$-valued solution $\rho(t)$, $t \geqslant 0$ and the map $\rho \to \sigma_t(\rho) = \rho(t)$, $t \geqslant 0$ is a one parameter contraction semigroup in the Banach space $(\mathscr{T}_h, \|\cdot\|_{\mathrm{tr}})$. On the other hand by Theorem 2.2 (i) there exists a unique regular $(\mathscr{H}_0, \mathscr{M})$-adapted contractive operator valued process $V \equiv \{V(t), t \geqslant 0\}$ satisfying

$$\mathrm{d}V(t) = \sum_{k \in S} V(t) Z^i_j \Lambda^j_i(t), \quad V(0) = I \tag{18}$$

on $\mathscr{H}_0 \underline{\otimes} \varepsilon(\mathscr{M})$ where

$$Z^i_j = \begin{cases} S^i_j - \delta^i_j, & i, j \in S, \\ Z_i, & i \in S, j = 0, \\ -\sum_{k \in S} Z_k^* S^k_j, & i = 0, j \in S, \\ Y, & i = 0, j \end{cases} \tag{19}$$

and $S=((S^i_j))$ is a contractive operator in $\mathscr{H}_0 \otimes l_2(S)$. The contractive one parameter semigroup $\tau_t := \mathbb{E}_0[V(t)^*(x\otimes I)\, V(t)]$, $t \geqslant 0$ of completely positive maps [1] and σ_t, $t \geqslant 0$ satisfy the relation

$$\operatorname{tr}(x\sigma_t(\rho)) = \operatorname{tr}(\rho\tau_t(x))$$

whenever $t \geqslant 0$, $\rho \in \mathscr{T}_h$, $x \in \mathscr{B}(\mathscr{H}_0)$.

Here our aim is to deal with the dilation problem associated with the Fokker-Planck equations (16)–(17) when the operators Y, Z_k, $k \in S$ are not necessarily bounded operators.

DEFINITION 4.1.

[9, 15] A one parameter family of completely positive maps $\tau \equiv \{\tau_t, t \geqslant 0\}$ on $\mathscr{B}(\mathscr{H}_0)$ is said to be a *quantum dynamical semigroup* if the following hold:

(i) $\tau_0(x) = x, \quad \tau_t(\tau_s(x)) = \tau_{s+t}(x), \quad s, t \geqslant 0, \quad x \in \mathscr{B}(\mathscr{H}_0)$;
(ii) $\|\tau_t\| \leqslant 1, t \geqslant 0$;
(iii) The map $t \to \operatorname{tr}(\rho\tau_t(x))$ is continuous for any fixed $x \in \mathscr{B}(\mathscr{H}_0)$ and $\rho \in \mathscr{T}$, the trace class operators in $\mathscr{H}_0$.
(iv) For each $t \geqslant 0$ the map $x \to \tau_t(x)$ is continuous in the ultra-weak operator topology.

Given a dynamical semigroup τ we define the predual semigroup $\sigma \equiv \{\sigma_t, t \geqslant 0\}$ on $\mathscr{T}$ as

$$\operatorname{tr}(x\sigma_t(\rho)) = \operatorname{tr}(\rho\tau_t(x)) \tag{20}$$

wherever $t \geqslant 0, \rho \in \mathscr{T}, x \in \mathscr{B}(\mathscr{H}_0)$. Note that the family σ is uniquely determined if (20) holds for $\rho := |f\rangle\langle g|$, $f, g \in \mathscr{H}_0$. It is also evident that σ is a strongly continuous one parameter semigroup in the Banach space $(\mathscr{T}, \|\cdot\|_{\text{tr}})$. Conversely, for a strongly continuous one parameter semigroup σ on $\mathscr{T}$, (20) determines a unique dynamical semigroup τ. Moreover for any $t \geqslant 0$, $\operatorname{tr}\sigma_t(\rho) = \operatorname{tr}(\rho)$, $\rho \in \mathscr{T}_h$ if and only if $\tau_t(I) = I$.

The central aim of this section is to exploit the theory developed in §3 and the construction of the minimal quantum dynamical semigroup, as outlined in Davies [4], in dilating the minimal semigroup in a boson-Fock space.

Before we proceed to the next result we state the following simple but useful lemmas without proof.

Lemma 4.2. *Let* $s.\lim_{n\to\infty} A_n = A$ *and* $s.\lim_{n\to\infty} B_n = B$. *Then* $\lim_{n\to\infty} A_n \rho B_n^* = A\rho B^*$ *in* $\|\cdot\|_{\text{tr}}$ *topology whenever* $\rho \in \mathscr{T}$.

Lemma 4.3. *Let* A_k, $k \geqslant 1$ *and* B_k, $k \geqslant 1$ *be two families of bounded operators such that both the series* $\Sigma_{k\geqslant 1} A_k^* A_k$ *and* $\Sigma_{k\geqslant 1} B_k^* B_k$ *converge in strong operator topology. Then for each* $\rho \in \mathscr{T}_h$ *the series* $\Sigma_{k \geqslant 1} B_k \rho A_k^*$ *converges in* $\|\cdot\|_{\text{tr}}$ *norm topology.*

As in Davies [4], let Y be the generator of a strongly continuous contractive semigroup in $\mathscr{H}_0$ and let Z_k, $k \in S$ be a family of densely defined operators on $\mathscr{H}_0$ such that

$$\mathscr{D}(Y) \subseteq \mathscr{D}(Z_k), \quad k \in S \tag{21}$$

and

$$\langle f, Yf\rangle + \langle Yf, f\rangle + \sum_{k\in S} \langle Z_k f, Z_k f\rangle \leqslant 0 \tag{22}$$

for all $f \in \mathscr{D}(Y)$.

In view of Lemma 4.2 the following relation

$$\kappa_t(\rho) = e^{tY}\rho e^{tY^*}$$

defines a strongly continuous, positive, one parameter, contraction semigroup on $\mathscr{T}_h$, whose generator G is given formally by

$$G(\rho) = Y\rho + \rho Y^*. \tag{23}$$

We introduce the positive one-to-one map π on $\mathscr{T}_h$ defined by

$$\pi(\rho) = (1-Y)^{-1}\rho(1-Y^*)^{-1}.$$

As in [4] we set $\pi(\mathscr{T}_h) = \{\pi(\rho),\ \rho \in \mathscr{T}_h\}$ and define the positive linear map $\mathscr{J}: \pi(\mathscr{T}_h) \to \mathscr{T}_h$ by

$$\mathscr{J}(\rho) = \sum_{k\in S} Z_k \rho Z_k^* \tag{24}$$

where the convergence follows from (22) and Lemma 4.3.

PROPOSITION 4.4.

Consider the family $Y, Z_k, k \in S$ of operators satisfying (21) *and* (22). *Then the following hold*:

(i) *$\pi(\mathscr{T}_h)$ is a core for G and* (23) *is valid for all $\rho \in \pi(\mathscr{T}_h)$*;

(ii) *The map $\mathscr{J}$ has a positive extension $\mathscr{J}'$ on $\mathscr{D}(G)$ such that*

$$tr(G(\rho) + \mathscr{J}'(\rho)) \leqslant 0 \tag{25}$$

wherever $\rho \in \mathscr{D}(G)$. Moreover equality holds in (25) *if and only if equality holds in* (22);

(iii) *For each fixed $\lambda > 0$, $\mathscr{J}'(\lambda - G)^{-1}$ is a map from $\pi(\mathscr{T}_h)$ into $\mathscr{T}_h$ and has a unique bounded positive extension A_λ in $\mathscr{T}_h$ such that $\|A_\lambda\| \leqslant 1$ and $\mathscr{J}'(\rho) = A_1[1-G](\rho)$ for all $\rho \in \mathscr{D}(G)$*;

(iv) *For any fixed $0 \leqslant r < 1$, $\pi(\mathscr{T}_h)$ is a core for the operator $W^{(r)} = G + r\mathscr{J}'$ defined on $\mathscr{D}(G)$. Moreover $W^{(r)}$ is the generator of a strongly continuous positive one parameter contraction semigroup $\sigma_t^{(r)}$, whose resolvent at $\lambda > 0$ is given by*

$$R_r(\lambda) \equiv (\lambda - W^{(r)})^{-1} = (\lambda - G)^{-1} \sum_{k\geqslant 0} r^k A_\lambda^k, \tag{26}$$

where the series converges in trace norm;

(v) *For each $\rho \geqslant 0$, $t \geqslant 0$ the map $r \to \sigma_t^{(r)}(\rho)$, $r \in [0,1)$ is increasing and continuous*;

(vi) *There exists a positive one parameter strongly continuous contraction semigroup σ_t^{min} on $\mathscr{T}_h$ such that*

$$\lim_{r\uparrow 1} \sigma_t^r(\rho) = \sigma_t^{min}(\rho)$$

for all $\rho \in \mathscr{T}_h$;

(vii) *For each* $\lambda > 0$, $R^{(n)}(\lambda) := (\lambda - G)^{-1} \sum_{0 \leq k \leq n} A_\lambda^k \to R(\lambda)$ *strongly as* $n \to \infty$, *where* $R(\lambda) = (\lambda - W)^{-1}$, *W is the generator of* σ_t^{min}.

Proof. For (i)–(vi) see Davies [4]. Now for (vii) we follow Kato [14] (Lemma 7). For each $\lambda > 0$, $0 \leq r < 1$ we have

$$R_r^{(n)}(\lambda) := (\lambda - G)^{-1} \sum_{0 \leq k \leq n} r^k A_\lambda^k \leq R_r(\lambda) \leq R(\lambda).$$

Letting $r \uparrow 1$ we get $R^{(n)}(\lambda) \leq R(\lambda)$. But as $R^{(n)}(\lambda)$ is increasing with n, $\text{s.lim}_{n \to \infty} R^{(n)}(\lambda) = R'(\lambda)$ exists and $R'(\lambda) \leq R(\lambda)$. We also have $R_r^{(n)}(\lambda) \leq R^{(n)}(\lambda) \leq R'(\lambda)$. Hence $R_r(\lambda) = \lim_{n \to \infty} R_r^{(n)}(\lambda) \leq R'(\lambda)$, $R(\lambda) = \lim_{r \uparrow 1} R_r(\lambda) \leq R'(\lambda)$ by (vi). This completes the proof.

Now our aim is to obtain a necessary and sufficient condition for σ to be trace preserving. It is evident that equality in (22) is necessary. We have the following theorem giving sufficient conditions.

Theorem 4.5. *Consider the semigroup* σ_t^{min}, $t \geq 0$ *defined as in Proposition* 4.4. *Let* $W_0 = G + \mathscr{J}'$ *with domain* $\pi(\mathscr{T}_h)$ *and let* W_0^* *be the adjoint of* W_0. *Assume furthermore the equality in* (22). *Then the following statements are equivalent:*

(i) $tr(\sigma_t^{min}(\rho)) = tr(\rho)$ *for all* $t \geq 0$, $\rho \in \mathscr{T}_h$;
(ii) *for each fixed* $\lambda > 0$, $A_\lambda^n \to 0$ *strongly as* $n \to \infty$;
(iii) *for each fixed* $\lambda > 0$, $(\lambda - W_0)(\pi(\mathscr{T}_h))$ *is dense in* $\mathscr{T}_h$;
(iv) *for each fixed* $\lambda > 0$, *the characteristic equation* $W_0^*(x) = \lambda x$ *has no non-zero solution in* $\mathscr{B}(\mathscr{H}_0)$;
(v) *for any fixed* $\lambda > 0$,

$$\beta_\lambda \equiv \{x \geq 0, x \in \mathscr{B}(\mathscr{H}_0) : \langle f, xYg \rangle + \langle Yf, xg \rangle + \sum_{k \in S} \langle Z_k f, x Z_k g \rangle = \lambda \langle f, g \rangle \tag{27}$$

hold for all $f, g \in \mathscr{D}(Y)\} = \{0\}$.

Proof. The proof is exactly along the lines of Theorem 3 in [14]. We write $\sigma = \sigma^{\min}$. As in [14] in this context we note that

$$\| R(\lambda)(\rho) \|_{\text{tr}} = \int_0^\infty \exp(-\lambda t) \| \sigma_t(\rho) \|_{\text{tr}} \mathrm{d}t \tag{28}$$

for all $\rho \geq 0$, which follows from the resolvent formula $R(\lambda) = \int_0^\infty \exp(-\lambda t) \sigma_t \mathrm{d}t$, $\lambda > 0$. As a simple consequence of the following identity

$$I + \mathscr{J}' R^{(n)}(\lambda) = (\lambda I - G) R^{(n)}(\lambda) + A_\lambda^{n+1} \tag{29}$$

and (22) and (25), we get $\text{tr}(\rho) = \lambda \, \text{tr}(R^{(n)}(\lambda)(\rho)) + \text{tr}(A_\lambda^{n+1}(\rho))$ for $\rho \in \mathscr{T}$. Since $R^{(n)}(\lambda)(\mathscr{T}_+) \subset \mathscr{T}_+$ we have

$$\| \rho \| = \lambda \| R^{(n)}(\lambda)(\rho) \|_{\text{tr}} + \| A_\lambda^{n+1}(\rho) \|_{\text{tr}} \tag{30}$$

for all $\rho \geqslant 0$. Now taking limit as $n \to \infty$ in (30) we get by Proposition 4.4(vii)

$$\lim_{n \to \infty} \| A_\lambda^{n+1}(\rho) \|_{\mathrm{tr}} = \| \rho \| - \lambda \| R(\lambda)(\rho) \|_{\mathrm{tr}}$$

$$= \lambda \int_0^\infty \exp(-\lambda t)(\| \rho \|_{\mathrm{tr}} - \| \sigma_t(\rho) \|_{\mathrm{tr}}) \tag{31}$$

for all $\rho \geqslant 0$, where we have used (28) in the second equality. Since for each fixed $\rho \in \mathscr{T}$ the map $t \to \| \sigma_t(\rho) \|_{\mathrm{tr}}$ is continuous and $\| \sigma_t(\rho) \|_{\mathrm{tr}} \leqslant \| \rho \|_{\mathrm{tr}}$, $t \geqslant 0$ from (31) we conclude that (i) and (ii) are equivalent.

Our next aim is to show that (ii) and (iii) are equivalent for any fixed $\lambda > 0$. From (29) we note that (ii) is equivalent to

$$\lim_{n \to \infty} [\lambda - G - \mathscr{J}'] R^{(n)}(\lambda)(\rho) = \rho$$

for all $\rho \in \mathscr{T}$. Since $R^{(n)}(\lambda)(\rho) \in \mathscr{D}(G)$ we conclude that $[\lambda - G - \mathscr{J}'](\mathscr{D}(G))$ is dense in $\mathscr{T}$. Since $\pi(\mathscr{T}_h)$ is a core for G, for any fixed $\rho \in \mathscr{D}(G)$ we choose a sequence $\rho_n \in \pi(\mathscr{T}_h)$ such that $\rho_n \to \rho$ and $G(\rho_n) \to G(\rho)$ as $n \to \infty$. By Proposition 4.4(iii) we have

$$\| \mathscr{J}'(\rho) \|_{\mathrm{tr}} = \| A_1 [1 - G)](\rho) \|_{\mathrm{tr}} \leqslant \| [1 - G](\rho) \|_{\mathrm{tr}} \leqslant \| \rho \|_{\mathrm{tr}} + \| G(\rho) \|_{\mathrm{tr}}$$

for all ρ in $\mathscr{D}(G)$, hence $\mathscr{J}'(\rho) = \lim_{n \to \infty} \mathscr{J}'(\rho_n)$. Thus it is evident that

$$(\lambda - G - \mathscr{J}')(\rho) = \lim_{n \to \infty} (\lambda - G - \mathscr{J}')(\rho_n)$$

Hence $[\lambda - G - \mathscr{J}'](\pi(\mathscr{T}_h))$ is dense in $\mathscr{T}_h$.

Conversely let (iii) be valid. Since $[I - A_\lambda](\mathscr{T}_h) = [I - A_\lambda][\lambda - G](\mathscr{D}(G)) = [\lambda - G - \mathscr{J}'][\pi(\mathscr{D}(G)] \supset [\lambda - G - \mathscr{J}'][\pi(\mathscr{T}_h)]$, $[I - A_\lambda](\mathscr{T}_h)$ is dense in $\mathscr{T}_h$. Set $C_\lambda^{(n)} = (1/n + 1) \Sigma_{0 \leqslant k \leqslant n} A_\lambda^k$, which is a uniformly bounded by 1. That $\mathrm{slim}_{n \to \infty} C_\lambda^{(n)} = 0$ is now an easy consequence of $C_\lambda^{(n)}[I - A_\lambda] = (1/n + 1)[I - A_\lambda^{n+1}]$. On the other hand, A_λ being a contractive positive map, $\| A_\lambda^m \| \leqslant \| A_\lambda^n \|$ whenever $m \geqslant n$, hence

$$\| C_\lambda^{(n)}(\rho) \|_{\mathrm{tr}} = \frac{1}{n+1} \sum_{0 \leqslant k \leqslant n} \| A_\lambda^k(\rho) \|_{\mathrm{tr}} \geqslant \| A_\lambda^n(\rho) \|_{\mathrm{tr}}$$

whenever $\rho \geqslant 0$. Thus we have $A^n(\rho) \to 0$ as $n \to \infty$. This shows that (ii) and (iii) are equivalent.

That (iii) and (iv) are equivalent follows by the definition of adjoint of a densely defined operator and Hahn–Banach theorem. Finally we need to show that an element $x \in \mathscr{D}(W_0^*)$ satisfies $W_0^*(x) = \lambda x$ if and only if x satisfies (27). For any fixed $f, g \in \mathscr{H}_0$ and $x \in \mathscr{D}(W_0^*)$ we have

$$\begin{aligned} \mathrm{tr}(\pi(|f><g|)\, W_0^*(x)) = \sum_{k \in S} \langle Z_k(1-Y)^{-1} f, x Z_k (1-Y)^{-1} g \rangle \\ + \langle Y(1-Y)^{-1} f, x(1-Y)^{-1} g \rangle \\ + \langle (1-Y)^{-1} f, x Y(1-Y)^{-1} g \rangle. \end{aligned} \tag{32}$$

Since $\mathscr{R}(1-Y)^{-1})=\mathscr{D}(Y)$, that (iv) and (v) are equivalent is a simple consequence of (32). ■

We use the same symbol for the linear canonical extension of a bounded map that appeared in Proposition 4.4 to the Banach space of all trace class operators. In the case of an unbounded operator, say G we extend it to $\mathscr{D}(G)+i\mathscr{D}(G)$ by linearity. The family of maps $\tau_t^{\min}:=(\sigma_t^{\min})^*$ on the dual space $\mathscr{B}(\mathscr{H}_0)$ is called the *minimal dynamical semigroup*. For further details we refer to [4].

Our next aim is to deal with the dilation problem associated with the Fokker-Planck equation (16) whenever the operators Y, Z_k, $k\in S$ satisfy the following assumption.

Assumption A. Y is the generator of a strongly continuous semigroup on $\mathscr{H}_0$ and Z_k, $k\in S$ is a family of densely defined operators satisfying (21) and (22). There exists a dense linear manifold $\mathscr{D}$ in $\mathscr{H}_0$ so that it is a core for Y and

$$S_j^k(\mathscr{D})\subseteq\mathscr{D}(Z_k^*),\quad k,j\in S$$

where $S\equiv((S_j^k,\ k,j\in S))$ is a contractive operator on $\mathscr{H}_0\otimes l_2(S)$. Furthermore for any fixed $j\in S$, $S_j^i\neq 0$ for finitely many $i\in S$. The last hypothesis ensures that the third expression (19) is meaningful and is indeed verified for most applications [19].

For any $\lambda>0$ we define bounded operators Y_λ, Z_k^λ, $k\in S$ by

$$Y_\lambda=\lambda^2(\lambda-Y^*)^{-1}Y(\lambda-Y)^{-1},\quad Z_k^\lambda=\lambda Z_k(\lambda-Y)^{-1}$$

where boundedness of Z_k^λ, $k\in S$ follows from (22). Moreover for each $\lambda>0$, Y_λ, Z_k^λ, $k\in S$ satisfies (17), hence the series $\Sigma_{k\in S}(Z_k^\lambda)^*Z_k^\lambda$ converges in strong operator topology. On the other hand for each $g\in\mathscr{D}(Y)$ we have $Y_\lambda g\to Yg$ as $\lambda\to\infty$. Taking $f=(I-\lambda(\lambda-Y)^{-1})g$ in (22) we get

$$\|Z_k(I-\lambda(\lambda-Y)^{-1})g\|\leqslant 2\|(I-\lambda(\lambda-Y)^{-1}g\|\,\|Y(I-\lambda(\lambda-Y)^{-1})g\|$$

Hence $Z_k^\lambda g\to Z_k g$ as $\lambda\to\infty$ for all $g\in\mathscr{D}(Y)$ [5].

For any $\lambda>0, 0\leqslant r\leqslant 1$ we define bounded operators $Z(\lambda,r)\equiv\{Z_j^i(\lambda,r), i,j\in\bar{S}\}$ as in (19) with $Y, Z_k, k\in S$ replaced by $Y_\lambda, r^{1/2}Z_k^\lambda, k\in S$ respectively. So for each $0\leqslant r\leqslant 1$ and $\lambda>0$, $Z(\lambda,r)\in\mathscr{Z}_R^-\cap\tilde{\mathscr{Z}}_R^-$. We denote by $V^{(\lambda,r)}=\{V^{(\lambda,r)}(t), t\geqslant 0\}$ the unique regular $(\mathscr{H}_0,\mathscr{M})$ – adapted contractive process satisfying (18) with $Z(\lambda,r)$ as its coefficients.

We also define operators $Z(r)$ on $\mathscr{D}$ as in (19) with Z_k replaced by $r^{1/2}Z_k$, $k\in S$ and write $Z(\lambda,1)=Z(\lambda)$, $Z(1)=Z$. For each $0\leqslant r\leqslant 1$ it is evident that

$$\lim_{\lambda\to\infty} Z_j^i(r,\lambda)f=Z_j^i(r)f,$$

for all $i,j\in\bar{S}$, $f\in\mathscr{D}$.

PROPOSITION 4.6.

Consider the operators Y, Z_k, $k\in S$ satisfying Assumption A. Then the following hold:

(i) *For each $0\leqslant r\leqslant 1$, w.lim$_{\lambda\to\infty}$ $V^{(\lambda,r)}(t)=V^{(r)}(t)$ exists for all $t\geqslant 0$ and $V^{(r)}=\{V^{(r)}(t),$ $t\geqslant 0\}$ is the unique regular $(\mathscr{H}_0,\mathscr{M})$-adapted contractive operator valued process*

satisfying (18) on $\mathscr{D}\otimes\varepsilon(\mathscr{M})$ *with* $Z(r)\equiv\{Z^i_j(r),\, i,j\in\bar{S}\}$ *as its coefficients. Moreover* $V^{(r)}$ *is a strongly continuous contractive bar-cocycle*;
(ii) *For each* $t\geqslant 0$ *the map* $r\to V^{(r)}(t), 0\leqslant r\leqslant 1$ *is continuous is weak operator topology.*

Proof. For each $0\leqslant r\leqslant 1$, $\lambda>0$; $Z(\lambda,r)\in\mathscr{Z}^-_R\cap\mathscr{Z}^-_R$ and the triad $Z(r), Z(\lambda,r), \mathscr{D}$ satisfy (12) and (13) with n replaced by λ. By our hypothesis $\mathscr{D}$ is also a core for Y. Hence we conclude (i) by Proposition 3.1 (i)–(ii).

Choose $0\leqslant r,r_n\leqslant 1 (n\geqslant 1)$ such that $r_n\to r$ as $n\to\infty$. Since the triad $(Z(r), Z(r_n), n\geqslant 1,\mathscr{D})$ satisfies (12) and (13) on $\mathscr{D}$ and $Z^i_j(r_n)f\to Z^i_j(r)f$, as $r_n\to\infty$ for any $f\in\mathscr{D}$, remark 3.2 implies that $w\lim_{n\to\infty} V^{(r_n)}(t)=V^{(r)}(t), 0\leqslant t<\infty$. This completes the proof. ■

For each $\lambda,\mu>0$, $0\leqslant r,s\leqslant 1$, we define the semigroup $\tau^{(\lambda,\mu,r,s)}$ on $\mathscr{B}(\mathscr{H}_0)$ by

$$\tau_t^{(\lambda,\mu,r,s)}(x)=\mathbb{E}_0[V^{(\lambda,r)}(t)^* x V^{(\mu,s)}(t)],\quad t\geqslant 0,$$

where the semigroup property follows from the cocycle property of the contractive processes $V^{(\lambda,r)}$. The associated pre-dual semigroup $\sigma^{(\lambda,\mu,r,s)}$ on $\mathscr{T}$ is defined as in (20) whose bounded generator $\mathscr{L}_*^{(\lambda,\mu,r,s)}$ is given by

$$\mathscr{L}_*^{(\lambda,\mu,r,s)}(\rho)=Y_\mu\rho+\rho Y^*_\lambda+\sqrt{rs}\sum_{k\in S}Z^\mu_k\rho(Z^\lambda_k)^*,\quad \rho\in\mathscr{T}.$$

For each $0\leqslant r<1$ we also have

$$W^{(r)}(\rho)=Y\rho+\rho Y^*+r\sum_{k\in S}Z_k\rho Z^*_k,\quad \rho\in\pi(\mathscr{T}),$$

where $W^{(r)}$ is described in Proposition 4.4.

We write $r^{(\lambda,r)}$, $\tau^{(\lambda,\mu,r)}$, $\sigma^{(\lambda,r)}$ and $\sigma^{(\lambda,\mu,r)}$ for $\tau^{(\lambda,\lambda,r,r)}$, $\tau^{(\lambda,\mu,r,r)}$, $\sigma^{(\lambda,\lambda,r,r)}$ and $\sigma^{(\lambda,\mu,r,r)}$ respectively. When $r=1$ we omit the symbol r. For each $0\leqslant r,s\leqslant 1$ we also define the one parameter semigroup

$$\tau_t^{(r,s)}:=\mathbb{E}_0[V^{(r)}(t)^* x V^{(s)}(t)],\quad t\geqslant 0$$

on $\mathscr{B}(\mathscr{H}_0)$. Again when $r=s=1$ we omit the symbol r.

Our aim is to show that $\sigma_t^{(\min)}$ is the pre-dual map of τ_t for all $t\geqslant 0$, where $\sigma^{\min}$ is defined as in Proposition 4.4. For this we need the following lemma.

Lemma 4.7. *Let* $A_k, k\geqslant 1$ *and* $B_k, k\geqslant 1$ *be two families of bounded operators such that the series* $\Sigma_{n\geqslant 1}A^*_kA_k$ *converges in strong operator topology and* $s\lim_{n\to\infty}B_n=B$. *Then for each* $\rho\in\mathscr{T}_h$, $\lim_{m,n\to\infty}C(m,n)\equiv\lim_{m,n\to\infty}\Sigma_{k\in S}A_kB_m\rho B^*_nA^*_k=\Sigma_{k\in S}A_kB\rho B^*A^*_k\equiv C$ *in* $\|\cdot\|_{tr}$ *norm.*

Proof. Lemma 4.3 implies that C, $C(m,n)$, $m,n\geqslant 1$ are elements in $\mathscr{T}$. For any fixed $m,n\geqslant 1$ and $\rho\in\mathscr{T}$ we have

$$\|C(m,n)-C\|_{\mathrm{tr}}\leqslant\sum_{k\geqslant 1}\{\|A_k(B_m-B)\rho(A_kB_n)^*\|_{\mathrm{tr}}$$
$$+\|A_kB\rho(A_k(B_n-B))^*\|_{\mathrm{tr}}.\}.$$

Hence for $\rho = |f\rangle\langle g|$ we have

$$\begin{aligned}\|C(m,n) - C\|_{\mathrm{tr}} &\leqslant \sum_{k\geqslant 1} \{\|A_k(B_m - B)f\| \, \|A_k B_n g\| + \|A_k Bf\| \, \|A_k(B_n - B)g\|\} \\ &\leqslant \left(\sum_{k\geqslant 1} \|A_k(B_m - B)f\|^2\right)^{1/2} \left(\sum_{k\geqslant 1} \|A_k B_n g\|^2\right)^{1/2} \\ &\quad + \left(\sum_{k\geqslant 1} \|A_k f\|^2\right)^{1/2} \left(\sum_{k\geqslant 1} \|A_k(B_n - B)g\|^2\right)^{1/2} \\ &\leqslant \alpha(\|(B_m - B)f\| \, \|B_n g\| + \|f\| \, \|(B_n - B)g\|) \leqslant \beta \|f\| \, \|g\|\end{aligned}$$

where α, β are some positive constants independent of f, g. Hence the result follows for $\rho = |f\rangle\langle g|$, $f, g \in \mathscr{H}_0$. For a general $\rho = \Sigma_i c_i |f_i\rangle\langle g_i|$, $\|f_i\| = \|g_i\| = 1$, $\Sigma_i |c_i| < \infty$, we use dominated convergence theorem to conclude the required result. ■

PROPOSITION 4.8.

Consider the family of operators $\{Y, Z_k, k \in S\}$ satisfying (21) *and* (22). *Then for each fixed $0 \leqslant r < 1$ the following hold:*

(i) *For each $\lambda, \mu > 0$, $0 \leqslant r, s \leqslant 1$,*

$$\sigma_t^{(\lambda,\mu,r,s)} = \sigma^{(\lambda,\mu,(rs)^{1/2})}, \quad t \geqslant 0;$$

(ii) *For $\rho \in \pi(\mathscr{T})$, $\lim_{(\lambda,\mu)\to\infty} \|\mathscr{L}_*^{(\lambda,\mu,r)}(\rho) - W^{(r)}(\rho)\|_{tr} = 0$ where the limit is independent of the order of λ, μ;*
(iii) *$\lim_{(\lambda,\mu)\to\infty} \|\sigma_t^{(\lambda,\mu,r)}(\rho) - \sigma_t^{(r)}(\rho)\|_{tr} = 0$ for all $\rho \in \mathscr{T}$, where $\sigma^{(r)}$ is the map defined as in Proposition* 4.4;
(iv) *The pre-dual map of $\tau_t^{(r)}$ is $\sigma_t^{(r)}$, $t \geqslant 0$;*
(v) *For each $0 \leqslant s < 1$, $\sigma_t^{(r,s)} = \sigma_t^{((rs)^{1/2})}$ for all $t \geqslant 0$.*

Proof. Since for each fixed $\lambda, \mu > 0$, $\mathscr{L}_*^{(\lambda,\mu,r,s)} = \mathscr{L}^{(\lambda,\mu,(rs)^{1/2}(rs)^{1/2})}$ we conclude (i) by the fact that a bounded generator uniquely determines the semigroup [5].

Now for (ii) first observe that

$$\mathscr{L}_*^{(\lambda,\mu,r)}(\pi(\rho)) = Y_\mu \pi(\rho) + \pi(\rho)\, Y_\lambda^* + r^2 \sum_{k\in S} Z_k^1(\mu(\mu - Y)^{-1})\rho(\lambda(\lambda - Y)^{-1})^*(Z_k^1)^*$$

and

$$W^{(r)}(\pi(\rho)) = Y\pi(\rho) + \pi(\rho)\, Y^* + r^2 \sum_{k\in S} Z_k^1 \rho (Z_k^1)^*$$

for all $\rho \in \mathscr{T}$ and $Y_\mu \pi(\rho) = \mu^2(\mu - Y^*)^{-1}(\mu - Y)^{-1}(Y(1 - Y)^{-1}\rho(1 - Y^*)^{-1})$. Now (ii) is immediate from Lemma 4.7.

Since $\pi(\mathscr{T})$ is a core for $W^{(r)}$ which is the generator of a strongly continuous contraction semigroup, (iii) is evident from (ii) and a standard result (Corollary 3.18 [5]) in the theory of semigroups.

For any fixed $f, g \in \mathscr{H}_0$, $\lambda, \mu > 0$ we have

$$\operatorname{tr}(x\sigma_t^{\lambda,\mu,r}(|f\rangle\langle g|)) = \langle fe(0), V^{(\lambda,r)}(t)^* x V^{(\mu,r)}(t) ge(0)\rangle$$

Hence (iv) follows from Proposition 4.6(i) and (iii). Finally, we arrive at (v) from (i) and (iii).

The following theorem establishes the main result.

Theorem 4.9. *Let $Y, Z_k, k \in S$ be a family of operators satisfying Assumption A. Consider the family $Z \equiv \{Z^i_j, i, j \in S\}$ defined as in* (19) *on $\mathscr{D}$. Then there exists a unique regular $(\mathscr{D}, \mathscr{M})$-adapted contractive process $V \equiv \{V(t), t \geqslant 0\}$ satisfying* (18) *on $\mathscr{D} \otimes \varepsilon(\mathscr{M})$. Moreover the following hold:*
(i) *$\tau_t^{\min}(x) = \mathbb{E}_0[V(t)^* x V(t)]$, where $\tau^{\min}$ is the minimal dynamical semigroup on $\mathscr{B}(\mathscr{H}_0)$ associated with* (16) *and* (17)
(ii) *Assume furthermore that S is an isometry and the equality in* (22) *holds. Then $Z \in \mathscr{I}$. In such a case V is isometric if and only if $\beta_\lambda = 0$ for some $\lambda > 0$, where β_λ is defined as in Theorem* 4.5 (v).

Proof. The first part is a restatement of Proposition 4.6 (i) for $r = 1$.

In view of Proposition 4.8 it is evident that for all $0 \leqslant r < 1$,

$$\begin{aligned}\operatorname{tr}(\sigma_t^{((r)^{1/2})}(|f\rangle\langle g|)x) &= \lim_{s\uparrow 1} \operatorname{tr}(\sigma_t^{((rs)^{1/2})}(|f\rangle\langle g|)x) \\ &= \lim_{s\uparrow 1} \langle fe(0), V^{(r)}(t)^* x V^{(s)}(t) ge(0)\rangle \\ &= \langle fe(0), V^{(r)}(t)^* x V(t) ge(0)\rangle\end{aligned}$$

for any $f, g \in \mathscr{H}_0$. Now taking limit as $r \uparrow 1$ in the above identity we get the required identity for (i) by Proposition 4.4(vi).

That $Z \in \mathscr{I}$ is simple to verify. The 'only if' part of (ii) follows from (i) and Theorem 4.5. For the converse we appeal to Proposition 3.1 (iv). This completes the proof. ■

Now combining Corollary 3.3 and Theorem 4.9 we arrive at necessary and sufficient conditions for V to be co-isometric.

Theorem 4.10. *Consider the family $V \equiv \{V(t),\ t \geqslant 0\}$ of operators defined as in Theorem 4.9. Suppose the family $\{Y^*, Z_k, k \in S\}$ of operators also satisfy* (21), (22) *and $\tilde{D}$ is a core for Y^* so that $\tilde{\mathscr{D}} \subset \mathscr{D}(Z_k^*)$, $k \in S$. Assume further the equality in* (22) *and S is a co-isometry then $Z \in \tilde{\mathscr{I}}$. In such a case V is co-isometric if and only if $\tilde{\beta}_\lambda = 0$ for some $\lambda > 0$, where $\tilde{\beta}_\lambda$ defined in Corolary* 3.3 *is modified as β_λ was in the statement of Theorem* 4.5(*v*).

Proof. S being a contractive operator we observe that

$$\sum_{k \in S} \| L_k f \|^2 \leqslant \sum_{k \in S} \| Z_k f \|^2$$

for each $f \in \mathscr{D}(Z_k)$, $k \in S$, where $L_k = \Sigma_{l \in S}(S^l_k)^* Z_l$. Hence the family $\{Y^*, L_k, k \in S\}$ also

satisfy (21) and (22). Thus $\tilde{Z}\in\mathscr{X}^-(\tilde{\mathscr{D}})$. The proof is complete once we appeal to Corollary 3.3.

Example 4.11. Let L_k, $k\in S$ be a family of closed operators in $\mathscr{H}_0$ and Y be the generator of a contractive C_0 semigroup satisfying (21) and (22). For each $k\in S$ consider the polar decomposition $L_k = S_k|L_k|$, where S_k is the partial isometry with initial subspace as $\overline{\mathscr{R}(|L_k|)}$, hence $S_k^* L_k = |L_k|$. Now with $Z_k = L_k, S_j^k = \delta_j^k S_k$, define the family of operators $Z \equiv \{Z_j^i, i,j\in\bar{S}\}$ as in (19) on $\mathscr{D}(Y)$. It is evident that Assumption A is valid. In general it is difficult to verify if β_λ or $\tilde{\beta}_\lambda$ or both are trivial. However, when $|L_k|$, $k\in S$ is a family of commuting self-adjoint operators then $\tilde{\beta}_\lambda = 0$ for some (hence for all) $\lambda > 0$. For more explicit examples refer [19].

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Almost periodicity of some Jacobi matrices

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Abstract. We show that random Jacobi matrices are almost periodic whenever they have purely absolutely continuous spectrum having finitely many bands.

Keywords. Almost periodicity; random potential; Jacobi inversion.

1. Introduction

Studies on random Schrödinger operators acquired importance in view of their usefulness in understanding the properties of condensed matter systems. The theory is well developed both in the continuous and in the discrete settings and there are excellent reviews by Simon [14], Spencer [15], Carmona [2] on this subject, in addition the theory also appears in the book of Cycone *et al* [5]. In one dimension the theory is much sharper as the spectral properties of such operators are shown to have consequences on the nature of the random potential, see for example the Kotani theory on the determinicity of potentials having some absolutely continuous spectrum ([7] for the continuous case and Simon [13] for the discrete case.) Further it was shown by Kotani and Krishna [8] and Craig [4] that some random Schrödinger operators with purely absolutely continuous spectrum of a certain type are almost periodic of necessity. The route for showing such a result was through inverse spectral theory. Showing the almost periodicity of such random potentials involved setting up and solving the Dubrovin equation for some spectral parameters, Jacobi inversion on a Riemann surface etc., All this was done for the continuous case by Levitan [9] McKean and Moerbeke [11], McKean and Trubowitz [10].

In the discrete case (i.e. for Jacobi matrices) the inverse spectral theory exists, see Kac, Moerbeke [6], Moerbeke [17], Toda [16] for the periodic case and Carmona, Kotani [3] for the random case and the references therein. However while the existence of solutions for the inverse spectral problem is given in the above works it was only in [6] and [17] that the nature of the matrices so constructed is presented. Of necessity these turn out to be periodic matrices. In this paper we show that in the discrete setting given a random potential, with finite band absolutely continuous spectrum, it is almost periodic, in the sense that the support of the random potential consists only of almost periodic sequences. The theory extends to the case of infinitely many bands and will appear in [1].

We have organized this paper in four sections. In the first we set up the direct spectral theory and identify the spectral functions that will play a role in the subsequent

sections. In the second section we do the inverse theory and finally § 3 the Jacobi inversion is presented.

2. Direct-theory

We consider $\Omega = l^2_{\mathscr{R}}(\mathbb{Z}, 1/(1+|n|^2))$ and $\mathbb{B}$ the associated Borel σ-algebra. We consider a bimeasurable invertible transformation T on Ω whose action is $T\omega(n) = \omega(n+1)$ and consider a probability measure $\mathbb{P}$ on $(\Omega, \mathbb{B})$ such that it is invariant and ergodic with respect to T. Given $\omega \in \mathrm{Supp}\ \mathbb{P}$ we consider the operator q^ω of multiplication by ω on $l^2(\mathbb{Z})$ and consider the family of operators $H^\omega = \Delta + q^\omega, \Delta$ the discrete Laplacian. We assume further that $\omega \in \mathrm{Supp}\ \mathbb{P}$ implies that ω is bounded as a sequence in which case $\omega \to H^\omega$ will be a measurable self-adjoint operator valued map. Then the general theory [2] of such operators shows that there exist constant sets, Σ, Σ_{ac}, Σ_{sc} and Σ_{pp} such that the spectrum, the absolutely continuous, singularly continuous and pure point spectra are the above sets respectively for the operators H^ω a.e. ω.

The main theorem of the paper is the following.

Theorem 2.1. *Suppose $(\Omega, \mathbb{B}, \mathbb{P})$ is as above with $\mathbb{P}$ ergodic with respect to the shift T acting on Ω. Suppose that for the associated self-adjoint operators H^ω the spectrum is purely absolutely continuous and consists of finitely many bands. Then, any ω in Supp $\mathbb{P}$ is an almost periodic sequence.*

We present the proof of this theorem at the end of § 3.

Before we proceed further let us outline the strategy employed in proving the theorem. The information on the absolutely continuous spectrum, together with the zeros of the Green function for a given potential q gives us an expression for the Green function. From this we obtain an expression for the sum of the Weyl m-functions. Using the additional property that the imaginary parts of the m-functions agree on the absolutely continuous spectrum, we obtain expressions for the difference of the Weyl m-functions. The Weyl m-functions thus obtained are identified as the values, on the two sheets of a Riemann surface, of a meromorphic function and the poles of this meromorphic function are related, via the trace formula, to the potential q. These poles are then shown to be related to theta functions, from which almost periodicity follows.

We would like to emphasize here that unlike the continuous case, [8, 4], where the reflectionless property of the potentials is sufficient [4], it is not sufficient here. The reason is that the Dubrovin equation, that came for free in the continuous case, was the heart of the matter there and its analog is missing in the discrete case.

Henceforth we fix a ω and consider H^ω on $l^2(\mathbb{Z})$ and drop the superscript also, referring to q^ω as q and the associated sequence $\omega(n)$ as $q(n)$. We shall also write the limits $\lim_{\varepsilon \to 0} g_{\lambda + i\varepsilon}$ as simply g_λ for ease of writing and the appropriate definition should be clear from the context. Then from Weyl theory it is known that for $\lambda \in \mathbb{C}^+$, the difference equation, $(H - \lambda)u_\lambda = 0$, has two independent solutions u_1, u_2 such that $u_1(0) = 1, u_1(1) = 0, u_2(0) = 0, u_2(1) = 1$. One also has unique solutions $u_{\pm,\lambda}$ in $l^2(\mathbb{Z}^\pm)$ and there exist holomorphic functions $m_\pm(\lambda)$ such that $u_{\pm,\lambda}(n) = u_1(n) + m_\pm(\lambda) u_2(n)$. We also have

$$m_\pm = -\frac{u_\pm(\pm 1)}{u_\pm(0)} \tag{1}$$

which can equivalently be taken as their definition. Weyl theory also gives the expression for the green kernel of H, for $m \leqslant n$ in terms of the Wronskian $[u_+, u_-]$, as

$$g_\lambda(n,m) = (H-\lambda)^{-1}(n,m) = \frac{u_+(n)u_-(m)}{[u_+, u_-]} \tag{2}$$

and the definition extended by symmetry to $m \geqslant n$. An evaluation of the Wronskian leads to the expression

$$(H-\lambda)^{-1}(n,n) = \frac{(-1)}{m_{+,n}(\lambda) + m_{-,n}(\lambda) + \lambda - q(n)} \tag{3}$$

where $m_{\pm,n}(\lambda)$ are defined as in (1) by taking $T^n q$ instead of q. Let us recall the following relations for $m_\pm$ from [13].

$$m_{\pm,n}(\lambda) = -\frac{u_\pm(n \pm 1)}{u_\pm(n)} \tag{4}$$

and

$$m_{\pm,n}(\lambda) = q(n) - \lambda - (m_{\pm,(n\mp 1)})^{-1} \tag{5}$$

and in terms of the operators $H_{\pm,n}$ of H restricted to the subspaces $\{u \in l^2(\mathbb{Z}^\pm)\colon (u(n)=0\}$, we have

$$m_{\pm,n} = (H_{\pm,n} - \lambda)^{-1}(\delta_{n\pm 1}, \delta_{n\pm 1}). \tag{6}$$

From now on we retain the superscript ω and consider the set

$$A = \{\lambda \in \mathbb{R} : \mathbb{E}\{\ln|m_+^\omega(\lambda + i0)|\} = 0\}.$$

Then we have the following theorem from the theory of random Jacobi matrices. See Simon [14] (proofs of Theorems 1 and 2) for the proofs.

PROPOSITION 2.2.

Suppose A is as above with $|A| > 0$, then $\mathbb{E}^H_{\mathrm{ac}}(A) \neq 0$. If further A is an open interval, then $\mathbb{E}^H_{\mathrm{sing}}(A) = 0$.

Theorem 2.3. *Suppose the spectrum of H^ω contains the set A, then*

$$\lim \operatorname{Re} g_{\lambda + i\varepsilon}(n,n) = 0 \text{ a.e } \omega \text{ a.e., } \lambda \in A.$$

For a.e. pair $\{(\omega,\lambda)\}$ we have

$$\operatorname{Im} m_+^\omega(\lambda) = \operatorname{Im} m_-^\omega(\lambda) \textit{ and } \operatorname{Re}[m_+^\omega(\lambda) + m_-^\omega(\lambda) + \lambda - q(0)] = 0.$$

Further, $\{m_+^\omega(\lambda), q(0)\}$, a.e. $\lambda \in A$ determines $\{q^\omega(n), n \in \mathbb{Z}\}$ uniquely a.e. ω.

In fact we can also deduce as in [4] or [8] that

COROLLORY 2.4.

Suppose the spectrum of the Jacobi matrix H^ω is purely absolutely continuous and is

the union of closed intervals (finitely or infinitely many), then we have

$$\operatorname{Re} g^{\omega}_{\lambda+i0}(n,n) = 0 \; \forall \omega \text{ and } \forall n \in \mathbb{Z}, \lambda \in \sigma(H^{\omega}).$$

also

$$\operatorname{Im} g^{\omega}_{\lambda+i0}(n,n) = 0 \; \forall \omega \text{ and } \forall n \in \mathbb{Z}, \lambda \in \mathbb{R} \backslash \sigma(H^{\omega}).$$

3. Inverse theory

In this section we specify the spectral data that are uniquely associated to a given potential q. The necessary spectral data are the band edges λ_i of the absolutely continuous spectrum, the Dirichlet eigenvalues ξ_i of an appropriate half space problem and the $\pm$ valued variables σ_i specifying as to which half space problem the Dirichlet eigen values ξ_i correspond.

DEFINITION 3.1.

A set A is called reflectionless whenever, $\operatorname{Re} g_{\lambda+i0}(n,n) = 0$, $\forall n \in \mathbb{Z}$ and a.e. $\lambda \in A$. Then we have,

Lemma 3.2. *Suppose the spectrum Σ of H is a reflectionless set and is also the union of closed sets $[\lambda_{2i}, \lambda_{2i+1}]$, $i = 0, \ldots, N$, then, the following are valid for every $n \in \mathbb{Z}$.*

$$\operatorname{Re} g_{\lambda+i0}(n,n) = 0 \text{ on } \Sigma \text{ and } \operatorname{Im} g_{\lambda+i0}(n,n) = 0 \text{ on } \mathbb{R} \backslash \Sigma.$$

There is a unique zero $\xi_i(n)$ of $g_{\lambda}(n,n)$ in each of the intervals $[\lambda_{2i-1}, \lambda_{2i}]$, $i = 1, \ldots, N$.

Proof. The first part of lemma follows from the assumption that the spectrum is a reflectionless set and the vanishing of the imaginary part on the complement of the spectrum in $\mathbb{R}$ is easy to verify. As for the zeros of Green functions we note that if a zero exists it is unique, since the Green function is real analytic and is strictly increasing as a function of λ in each connected component of $\mathbb{R} \backslash \Sigma$. If there is a zero in $(\lambda_{2i-1}, \lambda_{2i})$ call it $\xi_i(n)$. Otherwise if g_{λ} is positive there then set $\xi_i(n)$ to be λ_{2i-1} and if it is negative set $\xi_i(n)$ to be λ_{2i}. This proves the lemma.

DEFINITION 3.3.

We consider Ω as in the previous section. We call a point q of Ω a reflectionless potential if the spectrum of $\Delta + q$ is a reflectionless set.

DEFINITION 3.4.

We define the hull $\mathbb{H}(q)$ of a reflectionless potential as the closure in the topology of Ω of $\{q(\cdot + n) : n \in \mathbb{Z} \text{ and } q \in \Omega\}$.

PROPOSITION 3.5.

If q is a reflectionless potential with spectrum Σ of the type mentioned in the above lemma. Then each q in the Hull $\mathbb{H}(q)$ is also a reflectionless potential with the same spectrum.

Proof. It is known that the functions $m_{\pm}^{(k)}$ converge uniformly in $\mathbb{C}^+$ whenever a sequence of potentials q_k converges to q in the topology of Ω, see [7]. The rest of the proof follows as in [8].

Remark 3.6. We note that recalling the equations (2, 3, 4, 5) for $g_\lambda(n,n)$, $m_{+,n}(\lambda)$, $m_{-,n}(\lambda)$, any zero $\xi_i(n)$ of $g_\lambda(n,n)$ in $[\lambda_{2i-1}, \lambda_{2i}]$ corresponds to a pole of $m_{+,n}(\lambda)$ (or $m_{-,n}(\lambda)$). Equivalently to an eigen value of $H_{+,n}$ (or $H_{-,n}$).

It is possible to reconstruct the Green function of the Jacobi matrix satisfying the conditions of Lemma (3.2) from the spectrum and the zeros of the Green function.

Theorem 3.7. *Suppose the spectrum of H satisfies the conditions of Lemma 3.2. Then, the data* $\{\lambda_i\}$ *and* $\{\xi_i(n)\}$ *are sufficient to recover the Green function* $g_z(n,n)$ *uniquely.*

Proof. Since $g_z(n,n)$ is Herglotz, its logarithm $\ln g_z(n,n)$ is also Herglotz. Further $0 \leqslant \operatorname{Im} \ln g_z \leqslant \pi$ in $\mathbb{C}^+$ and the Herglotz representation theorem gives the expression for $\ln g_z(n,n)$ as

$$\ln g_z(n,n) = a + bz + \frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{1}{(x-z)} - \frac{x}{1+x^2}\right)\operatorname{Im}\ln g_{x+i0}(n,n)\mathrm{d}x. \tag{7}$$

Since we have that

$$\operatorname{Im}\ln g_{x+i0}(n,n) = \tfrac{1}{2}\pi \text{ on } \Sigma$$

and

$$\operatorname{Im}\ln g_{x+i0}(n,n) = 0 \text{ or } \pi \text{ for } g_{x+i0} > 0 \text{ or } < 0.$$

Equation (7) becomes, after dropping the zero contribution from the integrand,

$$\ln g_z(n,n) = a + bz + \frac{1}{\pi}\left\{\sum_{i=0}\int_{\lambda_{2i}}^{\lambda_{2i+1}} + \sum_{i=1}\int_{\lambda_{2i-1}}^{\xi_i(n)} + \int_{\lambda_{2N+1}}^{\infty}\right\}\cdot$$
$$\times\left(\frac{1}{(x-z)} - \frac{x}{1+x^2}\right)\operatorname{Im}\ln g_{x+i0}(n,n)\mathrm{d}x. \tag{8}$$

Now using the values of the imaginary part of the logarithm from (8) and performing the integration we have that

$$\ln g_z(n,n) = a + bz + \left\{\sum_{i=0}\int_{\lambda_{2i}}^{\lambda_{2i+2}} + \frac{1}{2}\sum_{i=1}\int_{\lambda_{2i-1}}^{\xi_i(n)}\right\}\left(\frac{1}{(x-z)} - \frac{x}{1+x^2}\right)\mathrm{d}x$$
$$+\ln(\lambda_{2N+1-z}) + C(\lambda_{2N+1}) \tag{9}$$

where $C(\lambda_{2N+1})$ is a constant. Now the asymptotic expansion for $(g_z(n,n))^2$ requires that it behaves like $1/z^2$ at ∞ since the spectrum of H is bounded. Hence, we see that collecting the constant terms in the above equation together, the expression reduces to

$$\ln g_z(n,n) = \frac{1}{2}\sum_{i=0}^{2N+1}\ln\frac{(\lambda_{2i+1}-z)}{(\lambda_{2i}-\lambda)} + \sum_{i=1}^{N}\ln\frac{(\xi_i(n)-z)}{(\lambda_{2i-1}-\lambda)}. \tag{10}$$

PROPOSITION 3.8.

Suppose H satisfies the conditions of Lemma 3.2, then the potential has the following expression in terms of $\{\lambda_i, \xi_i(n)\}$

$$q(n) = \tfrac{1}{2}\lambda_0 + \tfrac{1}{2}\sum_{i=1}^{N}(\lambda_{2i-1} + \lambda_{2i} - 2\xi_i(n)). \tag{11}$$

Proof. Using the asymptotic expansion of $g_\lambda(n,n)$ as a power series and comparing the $(1/\lambda^2)$ terms from equation (2) on one hand and equation (4) on the other we get the above formula.

Now we define two functions R and P_n as follows

$$R(\lambda) = \prod_{i=0}^{2N+1}(\lambda_{2i} - \lambda)(\lambda_{2i+1} - \lambda) \text{ and } P_n(\lambda) = \prod_{i=1}^{N}(\xi_i(n) - \lambda). \tag{12}$$

Then in terms of R and P_n the expression for $g_\lambda(n,n)$ becomes in view of the equation (10)

$$g_\lambda(n,n) = \frac{P_n(\lambda)}{\sqrt{R(\lambda)}}. \tag{13}$$

The sign of the square root is determined and fixed according to the requirement that $g_\lambda(n,n)$ is positive in $(-\infty, \lambda_0)$.

At this stage we see that the absolutely continuous spectrum together with the Dirichlet eigenvalues gives the Green function uniquely. The knowledge of the Green function $g_\lambda(0,0)$ is actually sufficient to recover the potential when it is symmetric about zero. However when the potential is not symmetric we need to know either of $m_\pm(\lambda)$ and $q(0)$ is sufficient to recover the potential uniquely. We proceed to find the parameters that will determine $m_+(\lambda)$ uniquely.

We already know from equation (4) that the Green function is written in terms of $m_\pm(\lambda)$. We observe that the function

$$F(\lambda) = -g_\lambda(0,0)^{-1} = m_+(\lambda) + m_-(\lambda) + \lambda - q(0) \tag{14}$$

is Herglotz in $\mathbb{C}^+$ and has purely imaginary boundary values of Σ and is analytic except for poles in the complement of Σ. The poles are located precisely at the zeros of $g_\lambda(0,0)$. Since the function $m_+(\lambda), m_-(\lambda)$ are also Herglotz as can be seen from equation (6), it is clear that if we can find an expression for $m_+(\lambda) - m_-(\lambda)$ we can find m_+ and m_-. To this end we have

PROPOSITION 3.9.

Suppose q is a reflectionless potential with spectrum of the type given in Lemma 3.2, suppose further that m_+ and m_- have the same imaginary parts of the spectrum Σ of q and ξ_i is a pole of $m_\pm$ according as σ_i is ± 1. Then $m_\pm$ can be uniquely recovered from the knowledge of Σ, $\{\xi_i(0)$ and $\sigma_i\}$.

Proof. From equation (14) it is clear that $m_+(\lambda) + m_-(\lambda) + \lambda - q(0)$ is Herglotz and has boundary values a.e. on $\mathbb{R}$ and poles at $\xi_i(0)$. Also m_+, m_- both are analytic in

$\mathbb{C}^+$ and have non zero imaginary parts on Σ and zero imaginary parts on $\mathbb{R}\backslash\Sigma$. Therefore the function $G(\lambda)=(m_+-m_-)(\lambda)$ is analytic in $\mathbb{C}^\pm$. Also G has zero imaginary part on $\mathbb{R}$. Further from equation (14) it is clear that the poles of $F(\lambda)$ are simple and come precisely from those of m_+ or m_-. Therefore $G(\lambda)$ has the same poles on $\mathbb{R}$. Therefore it is a meromorpic function in $\mathbb{C}$ with simple poles at ξ_i. Then we use the relation (14) to compute the residues of the function G at the poles ξ_i we find that

$$G(\lambda)=\sum_{i\in I_+}\frac{C_i}{(\xi_i-\lambda)}-\sum_{i\in I_-}\frac{C_i}{(\xi_i-\lambda)} \tag{15}$$

where $I_\pm=\{i:\xi_i \text{ is a pole of } m_\pm\}$ and C_i are the residues of $F(\lambda)$ at the points ξ_i. Explicitly C_i's turn out to be

$$C_i=\frac{\sqrt{R(\xi_i(0))}}{P'(\xi_i(0))}$$

prime denoting the derivative with respect to λ. Now defining the function σ from $I_+\cup I_-$ to $\{+,-\}$ we can write the expression for $G(\lambda)$ as

$$G(\lambda)=\sum_{i=1}^{N}\frac{\sigma_i C_i}{(\xi_i-\lambda)} \tag{16}$$

since the difference of the left and right hand sides of the above equation is an entire function bounded on $\mathbb{C}$ and has the value 0 at infinity, hence vanishes identically by Liouville's theorem. From equations (15) and (14) we can write the expressions for $m_\pm$ as

$$m_\pm(\lambda)=\frac{1}{2}\left\{\left[\frac{\sqrt{R(\lambda)}}{P_0(\lambda)}-\lambda+q(0)\right]\pm\sum_{i=1}^{N}\frac{\sigma_i C_i}{(\xi_i(0)-\lambda)}\right\}. \tag{17}$$

Clearly since the poles of $m_\pm$ in $\mathbb{R}$ are the eigenvalues of $H_\pm$, we can collect all the above results into the following

Theorem 3.10. *Suppose we have the operator $H=\Delta+q$ with spectrum Σ, a reflectionles set of the type assumed in Lemma 3.2. Further suppose on Σ, $\operatorname{Im} m_+=\operatorname{Im} m_-$ a.e. Then corresponding to each set of points $\{\xi_i(0),\sigma_i(0)\}$, there is a unique potential q such that the spectrum of the associated H is purely absolutely continuous and equals Σ and further $\{\xi_i(0):\sigma_i(0)=\pm\}$ are precisely the eigen values of $H_\pm$. In this case the values of the potential at 0 is given by equation (5).*

Remark 3.11. The above theorem is valid if q, $\xi_i(0)$, $\sigma_i(0)$, $m_\pm$ are replaced by $T^n q$, $\xi_i(n)$, $m_{\pm,n}$ for each fixed $n\in\mathbb{Z}$.

For use in the next section we note that equations (17) and (6) imply that the following relations are valid

$$m_{+,n}(\lambda)=\frac{1}{2}\left\{\left[-\lambda+q(n)+\sum_{i=1}^{N}\frac{\sigma_i C_i}{(\xi_i(n)-\lambda)}\right]-\frac{\sqrt{R(\lambda)}}{P_n(\lambda)}\right\} \tag{18}$$

and

$$-\lambda + q(n) - m_{-,n}(\lambda) = \frac{1}{2}\left\{\left[-\lambda + q(n)\right] + \sum_{i=1}^{N} \frac{\sigma_i C_i}{(\xi_i(n) - \lambda)}\right] + \frac{\sqrt{R(\lambda)}}{P_n(\lambda)}\right\}. \quad (19)$$

We also note the relation

$$\frac{P_n(\lambda)}{P_0(\lambda)} = \frac{g_\lambda(n,n)}{g_\lambda(0,0)} = \phi_{+,n}(\lambda)\phi_{-,n}(\lambda) \quad (20)$$

where

$$\phi_{+,n}(\lambda) = \prod_{i=0}^{n-1} m_{+,i}(\lambda) \text{ and } \phi_{-,n}(\lambda) = \prod_{i=0}^{n-1} (\lambda - q(i) + m_{-,i}(\lambda)). \quad (21)$$

Clearly that at ∞ the behaviour of $\phi_{\pm,n}(\lambda)$ is given by

$$\phi_{\pm,n}(\lambda) \approx (\lambda)^{\mp n}. \quad (22)$$

4. Almost-periodicity

In the last section we showed that given the operator $H = \Delta + q$ with purely absolutely continuous spectrum we can recover the operator H uniquely under certain additional data given by theorem 3.10. Apriori it is not clear that there is any relation between the zeros $\xi_i(n)$ and $\xi_i(0)$ of the Green functions $g_\lambda(n,n)$ and $g_\lambda(0,0)$. In this section however we show that actually these zeros cannot vary independently for each n but infact need to move in a way to make the function $\Phi(n) = \Sigma\xi_i(n)$ almost periodic as n varies. As a consequence the potential which necessarily satisfies the 'trace' formula equation (11) also becomes almost periodic in n.

We recall the relations (18, 19 and 20), of the last section, for the functions $\phi_\pm$. It is clear that these can be thought of as the same function ϕ on the Riemann surface $\mathscr{R}$ associated with the function $\sqrt{R(\lambda)}$. We recall that $\mathscr{R}$ is constructed by taking two copies of the λ sphere slit along $[\lambda_{2i}, \lambda_{2i+1}]$, $i = 1, \ldots, N$ and joined appropriately to form a two sheeted branched cover, with branch points the λ_i's, $i = 2, \ldots, 2N+1$. The points on the spheres corresponding to the points at infinity of the plane are denoted as p_∞ and $p_{\infty'}$ respectively and fall on either sheet of $\mathscr{R}$. On $\mathscr{R}$ we consider closed paths α_i and β_i, $i = 1, \ldots, N$ forming crosscuts in such a way that α_i lies on the upper sheet and encloses $[\lambda_{2i}, \lambda_{2i+1}]$, $i = 1, \ldots, N$ and β_i starts at λ_1 goes to λ_{2i}, $i = 1, \ldots, N$ on the upper sheet crosses over to the lower sheet and returns to λ_1. These paths form a set of generators for the group of closed paths on $\mathscr{R}$. We also choose a point P_0 on $\mathscr{R}$, away from the branch points, the points p_∞ and $p_{\infty'}$ and the paths α_i and β_i, as a reference point where the value of the function $1/\sqrt{R(\lambda)}$ is chosen and fixed. Then the family $\{\lambda^m \mathrm{d}\lambda/\sqrt{R(\lambda)}\}$ $m = 0, \ldots, N-1$ forms a collection of N holomorphic differentials in terms of which we choose a basis,

$$\mathrm{d}\omega_m = \sum_{k=1}^{N} c_{mk} \frac{\lambda^{k-1}\mathrm{d}\lambda}{\sqrt{R(\lambda)}} \quad (23)$$

of holomorphic differentials for the vector space of holomorphic differentials on $\mathscr{R}$,

which necessarily has dimension N. The bais $\{d\omega_m\}$ is normalized so that the matrix T with entries.

$$T_{ij}=\int_{\alpha_j} d\omega_i,\quad T_{i,j+N}=\int_{\beta_j} d\omega_i,\quad 1\leqslant j,i\leqslant N$$

takes the form $[I,\tau]$, where I is the identity matrix and $i\tau$, of necessity, a symmetric nonpositive definite matrix. T is called the period matrix associated with the basis α_i, β_i and $d\omega_i$.

Now we consider the single valued function $\phi_n(\lambda)$ on $\mathscr{R}$ which takes values $\phi_{\pm,n}(\lambda)$ defined in equation (21) on the upper and lower sheets of $\mathscr{R}$. Clearly a point ξ_i in $(\lambda_{2i-1},\lambda_{2i})$ has unique point p_i over it in $\mathscr{R}$ for each value ± 1 of σ_i. Therefore the single valued function $\phi_n(\lambda)$ has poles at $p_{\infty'}$ of order n and zero of order n at p_∞ (by equation 22) and also has exactly N poles of order 1 at the points $p_i(0)$ and N zeros of order 1 at the points $p_i(n)$ by (20). The exact location of these points in terms of which sheet they belong to is unimportant in the subsequent discussion. Having identified the function $\phi_n(\lambda)$ as having the appropriate behaviour we shall use it to construct the differential

$$d\omega(n)=\frac{d\ln\phi_n(\lambda)}{d\lambda}d\lambda \tag{24}$$

with the property that it has poles at $p_\infty, p_{\infty'}$ with residues n and $-n$ simple poles at $p_i(n)$, $p_i(0)\, i=1,\ldots,N$ with residues $+1$ and -1 respectively. This differential will be crucial for us to obtain a relation between the points $p_i(n)$ and $p_i(0)$. To start with we have the relations, which are consequences of Cauchy's integral formula

$$\int_{\alpha_j} d\omega(n)=2\pi i k_j \text{ and } \int_{\beta_j} d\omega(n)=2\pi i m_j. \tag{25}$$

Now $d\omega(n)$ being a differential with poles we can write it in terms of the differentials of first, second and normalized differentials of the third kind as

$$d\omega(n)=D+nd\omega(p_\infty,p_{\infty'})+\sum_{i=1}^{N} d\omega(p_i(n),p_i(0))+\sum_{j=1}^{N} c_j d\omega_j \tag{26}$$

where $d\omega(a,b)$ is a normalized differential of the third kind with residues $+1$ and -1 respectively at a and b. D is a differential of the second kind and $d\omega_j$ are defined in (23). Since it is always possible to add differentials $d\omega_j$ to any $d\omega(a,b)$ to make the integral over the α_j vanish we obtain the relation that in view of the normalization (25) $c_j=2\pi i k_j$ for some integer k_j for each j by integrating the above equation over α_j.

Lemma 4.1. *We have the following relation among the points $p_i(n)$, $p_i(0)$, p_∞ and p_∞*

$$\sum_{j=1}^{N}\int_{p_0}^{p_j(n)} d\omega_i=\sum_{j=1}^{N}\int_{p_0}^{p_j(0)} d\omega_i-\sum_{j=1}^{N} k_j\tau_{ij}+m_i+\int_{p_\infty}^{p_\infty} d\omega_i \tag{27}$$

for each $i=1,\ldots,N$.

Proof. Integrating of equation (26) on both sides with respect to β_i we have

$$\int_{\beta_i} \mathrm{d}\omega(n) = +n\int_{\beta_i} \mathrm{d}\omega(p_\infty, p_{\infty'}) + \sum_{i=1}^{N}\int_{\beta_i} \mathrm{d}\omega(p_i(n), p_i(0)) + \sum_{j=1}^{N} c_j \int_{\beta_i} \mathrm{d}\omega_j. \tag{28}$$

By the comment before the lemma we know that the sum of integrals w.r.t. β_i of $\mathrm{d}\omega_j$ gives the term, $2\pi i\Sigma_{i=1}^{N} k_j \tau_{ij}$ and the integral of the left hand side provides us with the term $2\pi i m_i$. The remaining sum is computed as follows. Suppose $\mathrm{d}\omega(a,b)$ is a normalized differential of the third kind with residues $+1$ and -1 at p and q respectively. Then, we claim that

$$\int_{\beta_i} \mathrm{d}\omega(p,q) = 2\pi i\int_q^p \mathrm{d}\omega_l.$$

To show the claim we note first that by addition of differentials of first kind we can always make $\int_{\alpha_i}\mathrm{d}\omega(p,q)$ vanish for each i. Now consider the normalized differentials $\mathrm{d}\omega_k$ and compute the integral $\int_C \omega_k \mathrm{d}\omega(p,q)$, where ω_k is the integral of the differential $\mathrm{d}\omega_k$. Then $\omega_k \mathrm{d}\omega(p,q)$ is an Abelian differential, regular everywhere except for poles p, q. Therefore by Cauchy integral theorem the integral evaluates on the one hand to

$$\int_C \omega_k \mathrm{d}\omega(p,q) = 2\pi i(\omega_k(p) - \omega_k(q)) \tag{29}$$

since the residues of $\mathrm{d}\omega(p,q)$ at p and q are respectively $+1$ and -1. On the other hand going down to the polygonal region S corresponding to the normal form of the canonically dissected Riemann surface we obtain the relation (see Siegal [12], Chapter 4, § 7)

$$\int_C \omega_k \mathrm{d}\omega(p,q) = \sum_i \{\omega_k(A_i)\omega(p,q)(B_i) - \omega_k(B_i)\omega(p,q)(A_i)\} \tag{30}$$

where A_i, B_i are the sides of the polygon corresponding to the closed paths α_i, β_i. Now, we use the normalizations and their implications

$$\int_{\alpha_i} \mathrm{d}\omega(p,q) = 0 \Leftrightarrow \omega(p,q)(A_i) = 0 \text{ and } \int_{\alpha_i} \mathrm{d}\omega_k = \delta_{ik} \Leftrightarrow \omega_k(A_i) = \delta_{ik}. \tag{31}$$

Hence equation (29), and (31) together yield,

$$\omega(p,q)(B_k) = \int_C \omega_k \mathrm{d}\omega(p,q) = 2\pi i(\omega_k(p) - \omega_k(q)). \tag{32}$$

But the integral $\omega_k(p) - \omega_k(q)$ is precisely $\int_{P_0}^{p}\mathrm{d}\omega_k - \int_{P_0}^{q}\mathrm{d}\omega_k$ from which the claim follows.

Equation (27) can be inverted to get a relation for $P_j(n)$'s in terms of the right hand side of (27). The existence of a unique inverse is assured by the Jacobi inversion. Then an explicit formula will be obtained for the inverse through Θ functions. Therefore we define the necessary quantities here.

DEFINITION 4.2.

A divisor P is a formula product $P_1 \dots P_m/Q_1 \dots Q_k$ of points in $\mathscr{R}$. An integral divisor is the product of the type $P_1 \dots P_m$. An integral divisor $P_1 \dots P_N$ is called general if the matrix with entries $A_{ij} = (d\omega_i/d\lambda)(P_j), i, j = 1, \dots, N$ is nonsingular.

In the following theorem we identify the divisors which are general.

Lemma 4.3. *An integral divisor $p_1 \dots p_N$ is general provided the factors p_i are distinct and no two of the p_i is in $\{p_\infty, p_{\infty'}\}$.*

Proof. We note that the result follows by checking that the matrix J_{ij} with entries $(d/d\lambda)(\lambda^{(j-1)}/\sqrt{R(\lambda)})$ at P_j is nonsingular. Using the local parameters $t = z$ at ordinary points and $t = \sqrt{(z - P_i)}$ at branch points the verification is easy.

DEFINITION 4.4.

The Θ function on $\mathbb{C}^N$ is defined by

$$\Theta(z) = \sum_{m \in \mathbb{Z}^N} \exp(2\pi i \langle m, z \rangle) \exp(\pi i \langle m, \tau m \rangle) \tag{33}$$

where τ is as defined after equation (23).

The Θ function satisfies the periodicity relations as follows

$$\Theta(z + e_k) = \Theta(z) \text{ and } \Theta(z + \tau_k) = \exp[-2\pi i z_k - \pi i \tau_{kk}]\Theta(z). \tag{34}$$

The Jacobi's imaginary transformation is given by

$$\Theta(u, \tau) = \frac{i^{N/2}}{\sqrt{|\tau|}} \exp[-\pi i \langle u, \tau^{-1} u \rangle] \Theta(\tau^{-1} u, -\tau^{-1}) \tag{35}$$

where $\Theta(u, \tau)$ is the theta function at u with the period matrix given by τ. Next we consider a divisor $P = P_1 \dots P_N$ and define the Abel–Jacobi function $A(P)$ as the function $A(P) = \Sigma_{i=1}^N \int_{p_0}^{p_i} d\omega$, taking the divisor to a point in $\mathbb{C}^N/\Pi$. We recall the following theorems from Siegel [12], § 10.

Theorem 4.5. *Consider $A(P)$ defined above for an integral divisor P. Whenever P is general, then it is the unique integral divisor in the preimage of $A(P)$.*

While the above theorem guarantees a solution for the Jacobi inversion, the components of the point P are obtained as the zeros of an appropriate theta function acting on $\mathscr{R}$. Explicitly we consider on $\mathscr{R}$ the integrals $\omega(P) = \int_{P_0}^P d\omega$. Then we consider the function $\phi(P) = \Theta(\omega(P) - s + \tilde{c})$ where $\tilde{c}$ is a vector of Riemann constants depending upon τ and $\omega, s \in \mathbb{C}^N/\Pi$. It is also known that there are exactly N zeros for the function $\phi(P)$ in $\mathscr{R}$. We have the following theorem from [12] (§ 10).

Theorem 4.6. *Suppose $\phi(P)$ does not vanish identically for a fixed s. Then, the zeros $q_k \in \mathscr{R}$ of $\phi(P)$ satisfy the relation $s = \Sigma_{k=1}^N \int_{P_0}^{q_k} d\omega$, the equation is to be understood component wise.*

The upshot of the above theorems is that if we rewrite equation (27) as

$$\sum_{j=1}^{N} \int_{P_0}^{P_j(n)} d\omega_i = nc_i + K_i \tag{36}$$

to obtain $P_j(n)$ in terms of $nc + K$, one needs only to check that the required inverse is a general point and consider the function

$$\Theta\left(\omega(P) - \sum_{i=1}^{N} \int_{P_0}^{P_i(n)} d\omega + \tilde{c}\right) = \Theta(\omega(P) - nc - K + \tilde{c}) \tag{37}$$

and find its zero. The Riemann constant $\tilde{c}$ is chosen so that the function $\phi(P)$ does not vanish identically.

Till now we are still on the Riemann surface and identified the function whose zeros are precisely the points above $\xi_i(n)$. Now we shall obtain an expression for the sum of $\xi_i(n)$ in terms of the theta function mentioned above. To this end we have the following Lemmas. We consider the map λ from $\mathscr{R}$ to $\mathbb{C}$. The map λ is a meromorphic function on $\mathscr{R}$ with poles precisely at the points p_∞ and $p_{\infty'}$.

Lemma 4.7. *We have the following relation between the theta function and the* $\xi_i(n)$.

$$\sum_{i=1}^{N} \xi_i(n) = \text{Const} + \sum_{i=1}^{N} C_{iN} D_i \ln\left(\frac{\Theta(nc+d)}{\Theta((n+1)c+d)}\right) \tag{38}$$

where the constants $\{C_{iN}\}$ *are the same as those which appear in equation* (1), D_i *is the partial derivative in the ith direction,* **c** *is a vector with purely imaginary entries and* **d** *some constant vector independent of* **n**.

Proof. We consider the meromorphic differential $\lambda d \ln(\phi(P))$. It has poles at $p_\infty, p_{\infty'}$ and at the zeros $P_i(n)$ or $\phi(P)$. Let C be a closed contour issuing from the reference point P_0 and enclosing all the poles and zeros of the function $\lambda d \ln(\phi(\lambda))$. Therefore a computation using the Cauchy's integral theorem gives that

$$\int_C \lambda d \ln \phi = \sum_{i=1}^{N} \lambda(P_i(n)) + \text{Residue at } p_\infty + \text{Residue at } p_{\infty'}. \tag{39}$$

The computation of the residues is done using the local parameter as follows. The local parameter at p_∞ is $\lambda = \zeta^{-1}$. Therefore we have

$$\lambda \frac{d}{d\lambda} \ln(\phi(\omega(\lambda) - nc + d))|_{p_\infty} = \sum_{i=1}^{N} \frac{d}{d\lambda} \omega(\lambda) \cdot \{D_i \ln(\phi(\omega(\lambda) - (nc + d)))\}|_{p_\infty} \tag{40}$$

where ω is the integral of the differential $d\omega$ and D_i refers to the derivative of the argument of ϕ in the ith direction. The summand of the above equation is evaluated as

$$\frac{d}{d\lambda}\omega(\lambda) = -\frac{1}{\zeta^2} \sum_{j=1}^{N} C_{ij} \frac{\zeta^{-j+1}}{\sqrt{R(\zeta^{-1})}}. \tag{41}$$

Since p_∞ belongs to the upper sheet, the square root has a positive sign and also since $R(\lambda)$ goes like λ^{N+1} at ∞, we have that the above equation evaluates to $-C_{iN}$.

Therefore we have

$$\text{Residue at } p_\infty = -\sum_{i=1}^{N} C_{iN} D_i \ln(\Theta(\omega(p_\infty)) - nc + d). \tag{42}$$

Similarly noting that at $p_{\infty'}$, the square root has a negative sign since $p_{\infty'}$ belongs to the lower sheet of $\mathscr{R}$, we have

$$\text{Residue at } p_{\infty'} = \sum_{i=1}^{N} C_{iN} D_i \ln(\Theta(\omega(p_{\infty'})) - nc + d). \tag{43}$$

Now, since $c = \int_{p_\infty}^{p_\infty} \mathrm{d}\omega$, we have $\omega(p_\infty) = c + \omega(p_{\infty'})$. Putting these equations together and absorbing the value $\omega(p_\infty)$ into $\mathbf{d}$,

$$\int_C \lambda \mathrm{d} \ln \phi = \sum_{i=1}^{N} \xi_i(n) - \sum_{i=1}^{N} C_{iN} D_i \left\{ \ln \frac{\Theta(nc+d)}{\Theta((n+1)c+d)} \right\}. \tag{44}$$

This equation provides us with one expression for the left hand side. On the other hand going down to the Riemann region corresponding to the canonically dissected surface and taking the polygonal path $\{a_1 b_1 a_1^{-1} b_1^{-1} \dots b_N^{-1}\}$, we can integrate the function explicitly using the relations, for each k,

$$\frac{\mathrm{d}f}{f}(a_k^{-1}) = \frac{\mathrm{d}f}{f}(a_k) - 2\pi i \mathrm{d}\omega_k \text{ and } \frac{\mathrm{d}f}{f}(b_k^{-1}) = \frac{\mathrm{d}f}{f}(b_k). \tag{45}$$

The equations are understood to mean equality of evaluation of the differentials at a point on the ark a_k, b_k and the corresponding point on a_k^{-1}, b_k^{-1} etc., Then we obtain that

$$\int_C \lambda \mathrm{d} \ln \phi = \sum_{i=1}^{N} \int_{\alpha_i} \lambda \mathrm{d}\omega_i. \tag{46}$$

Putting equations (39) and (37) together we obtain the lemma.

Lemma 4.8. *The sum $\Sigma_{i=1}^{N} \xi_i(n)$ of the above lemma are almost periodic in n.*

Proof. The Jacobi-imaginary transformation equation (35) shows that the right hand side of the equation (38) for the sum $\Sigma_{i=1}^{N} \xi_i(n)$ is indeed real and further the vector $\tau^{-1} c$ is a vector with real entries. Therefore the required almost periodicity is now immediate from that of the theta function with real argument.

Now we are ready to prove the Theorem 1 of § 1.

Proof. The assumptions of Theorem 3.10 are satisfied for each fixed ω in support of $\mathbb{P}$. Therefore for each fixed ω the potential q^ω is almost periodic as a sequence in view of Lemma 4.8 and the trace formula.

We would like to make a few comments regarding the theorems of this paper. Firstly if we replace Δ in H by non constant off diagonal entries (i.e. coming from a positive sequence a_n), the theorems of the last two sections will go through with some modifications for the expressions for the m-functions and the trace formula. Secondly we have stated that the meromorphic function ϕ_n has exactly N poles. It might happen that if for some n, some of the zeros $\xi_i(n)$ coincide with $\xi_i(0)$, then this is not

exactly correct. However the proof still go through as this phenomenon does not persist for the neighbouring values $n-1$ and $n+1$. Such coincidence will show only the periodicity of the appropriate $\xi_i(n)$ in n.

A forteriori using the first order equations (4) for $m_+(\lambda)$, we can write a difference equations for the $\xi_i(n)$ in terms of $\xi_i(n-1)$ as follows. This equation shows clearly that the poles of $m_{+,n}$ are determined by the zeros of $m_{+,n-1}$ and similarly for m_-. Therefore by looking at the appropriate single valued function on the Riemann surface which agrees with m_+ on the upper and $\lambda - q + m_-$ on the lower sheets, we can write the following equation for the $p(\vec{n})$ above $\xi_i(\vec{n})$ as

$$\vec{p}(n) = V(\vec{p}(n), \vec{p}(n-1)). \tag{47}$$

The explicit form for V can be obtained by combining equations (4) and (18, 19). This equation will provide the analogue of the Dubrovin equation of the continuous case.

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Three-dimensional diffraction of compressional waves by a rigid cylinder in an inhomogeneous medium

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Abstract. In this paper we consider the diffraction of compressional waves by a rigid cylinder embedded in an unbounded inhomogeneous elastic medium. The point source, generating the incident pulse, is situated at a finite distance from the obstacle. It is assumed that the velocities of P and S waves are given by $\alpha = \alpha_0 r^q$, $\beta = \beta_0 r^q$ respectively, $q < 1$. The formal solutions of displacement field are obtained in the integral form. These integrals are evaluated asymptotically by the Residue Cagniard method to obtain the short-time estimate of the motion near the wave front in the shadow zone of the elastic medium. Numerical computations are done to investigate the behaviour of diffracted P and S waves.

Keywords. Diffraction; elastic inhomogeneity; residue-Cagniard method.

1. Introduction

Friedlander [3] applied the modified form of Fermat's principle to the problem of diffraction of two-dimensional pulses by a rigid cylinder in a homogeneous medium. Using Friedlander's method Jha and Mishra [5] solved the problem of diffraction of sound pulses by a rigid cylinder in an inhomogeneous medium. Kesari and Rajhans [6] studied the scattering of shear (SV) waves by a rigid cylinder in an inhomogeneous medium using Friedlander's method. Hwang *et al* [4] applied a similar method and discussed the case of three-dimensional elastic wave scattering by a rigid cylinder in an elastic medium.

In this paper we investigate the diffraction of compressional waves by a rigid cylinder in an inhomogeneous medium using the technique of Hwang *et al* [4]. The point source generating the incident pulse is supposed to be situated in the surrounding elastic medium at a finite distance from the obstacle. It is assumed that the velocities of P and S waves are functions of r only and are given by $\alpha = \alpha_0 r^q$ and $\beta = \beta_0 r^q$ respectively, where q is the inhomogenity factor and $0 < q < 1$. This law of variation of velocity has been found to be prevalent in actual earth ([1], [8]) and is more general as it includes the homogeneous case when $q = 0$.

The results show that in the case of a compressional point source there exist in the shadow zone (1) a diffracted PP_dP wave, that is, the P wave incident on the surface of the cylinder travels along the cylindrical surface as the P wave and finally emerges as the diffracted P wave, (2) a diffracted PS_dS wave, that is, the P wave incident on the surface is converted into the S wave which finally emerges as the diffracted S wave, (3) a diffracted PP_dS wave, which denotes that the P wave incident on the

surface travels on the cylindrical surface as the P wave and it emerges as the diffracted S wave after mode conversion ([1], [4]).

2. Formulation of the problem and formal solution

Let the axis of the cylinder be taken as the z axis and let the point source be located at $r = r_0$, $\theta = 0$, $z = 0$ (figure 1).

The velocity potentials φ and Ψ are functions of r, θ, z and t which satisfy the wave equations

$$\nabla^2\Phi - \frac{1}{\alpha^2}\ddot{\Phi} = -\frac{2\pi}{r}\delta(r-r_0)\delta(z)\delta(t)\sum_{m=-\infty}^{\infty}\delta(\theta+2m\pi) \quad (r_0 \geqslant r \geqslant a) \tag{1}$$

$$\nabla^2\Psi - \frac{1}{\beta^2}\ddot{\Psi} = 0 \qquad (r \geqslant a) \tag{2}$$

where

$$\alpha = \{(\lambda + 2\mu)/\rho\}^{1/2} \tag{3}$$

and

$$\beta = \sqrt{\mu/\rho}$$

The notations used in (1), (2), and (3) are defined as follows

∇^2: Laplacian operator
δ: Dirac delta function
λ, μ: Lame's parameters
ρ: Density of the medium
α, β: Velocities of P and S waves respectively.

Equation (2) can be divided into two parts namely [4]

(1) $$\nabla^2 G - \frac{1}{\beta^2}\ddot{G} = 0 \text{ and} \tag{4}$$

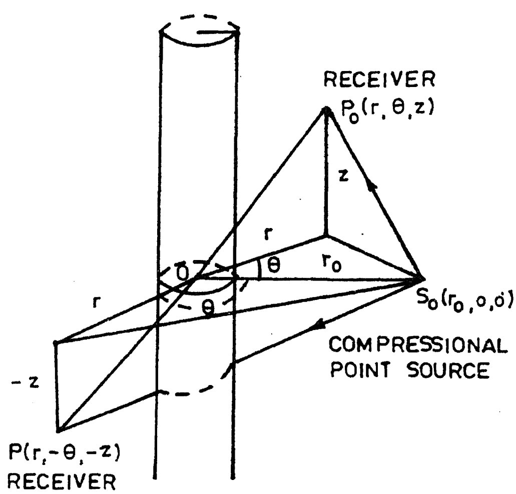

Figure 1. The geometry of the problem.

$$(2) \qquad \nabla^2 H - \frac{1}{\beta^2}\ddot{H} = 0, \tag{5}$$

where G and H are scalar wave functions.

The boundary conditions are

$$[U_r]_{r=a+0} = 0, \quad [U_\theta]_{r=a+0} = 0, \quad [U_z]_{r=a+0} = 0, \tag{6}$$

where U_r, U_θ and U_z are components of displacement.

We define Laplace transform pairs with respect to t as

$$\bar{\Phi}(r, \theta, z, s) = \int_{-\infty}^{\infty} \Phi(r, \theta, z, t)\exp(-st)\,\mathrm{d}t \tag{7}$$

$$\Phi(r, \theta, z, t) = \frac{1}{2\pi i}\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \bar{\Phi}(r, \theta, z, s)\exp(st)\,\mathrm{d}s, \tag{8}$$

where γ_1 lies to the right of all singularities in the s-plane. We define bilateral Laplace transform with respect to z as

$$\hat{\Phi}(r, \theta, p, s) = \int_{-\infty}^{\infty} \bar{\Phi}(r, \theta, z, s)\exp(-pzs)\,\mathrm{d}z \tag{9}$$

$$\bar{\Phi}(r, \theta, z, s) = \frac{1}{2\pi i}\int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} \hat{\Phi}(r, \theta, p, s)\exp(psz)\,\mathrm{d}(ps). \tag{10}$$

We further, define bilateral Laplace transform pairs with respect to θ as

$$\Phi^*(r, v, p, s) = \int_{-\infty}^{\infty} \hat{\Phi}(r, \theta, p, s)\exp(-v\theta)\,\mathrm{d}\theta \tag{11}$$

$$\hat{\Phi}(r, \theta, p, s) = \frac{1}{2\pi i}\int_{\gamma_3 - i\infty}^{\gamma_3 + i\infty} \Phi^*(r, v, p, s)\exp(v\theta)\,\mathrm{d}v. \tag{12}$$

Using these transformations in (1), (4) and (5) for $m = 0$ we get

$$\frac{\partial^2 \Phi^*}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi^*}{\partial r} - \left(f_1^2 - \frac{v^2}{r^2}\right)\Phi^* = -\frac{2\pi}{r}\delta(r - r_0) \quad (r_0 \geqslant r \geqslant a) \tag{13}$$

$$\frac{\partial^2 G^*}{\partial r^2} + \frac{1}{r}\frac{\partial G^*}{\partial r} - \left(f_2^2 - \frac{v^2}{r^2}\right)G^* = 0 \qquad (r \geqslant a) \tag{14}$$

$$\frac{\partial^2 H^*}{\partial r^2} + \frac{1}{r}\frac{\partial H^*}{\partial r} - \left(f_2^2 - \frac{v^2}{r^2}\right)H^* = 0 \qquad (r \geqslant a) \tag{15}$$

where

$$f_1^2 = \left(\frac{s^2}{\alpha_0^2 r^{2q}} - p^2 s^2\right), \quad f_2^2 = \left(\frac{s^2}{\beta_0^2 r^{2q}} - p^2 s^2\right).$$

Also the transformed boundary conditions are

$$U_r^* = \frac{\partial^2 \Phi^*}{\partial r^2} - \frac{v}{r}G^* + ps\frac{\partial H^*}{\partial r} = 0 \tag{16}$$

$$U_\theta^* = \frac{v}{r}\Phi^* - \frac{\partial G^*}{\partial r} + \frac{vps}{r}H^* = 0 \tag{17}$$

$$U_z^* = ps\Phi^* - \left(\frac{\partial^2 H^*}{\partial r^2} + \frac{1}{r}\frac{\partial H^*}{\partial r} - \frac{v^2}{r^2}H^*\right) = 0. \tag{18}$$

The solutions of the ordinary differential equations (13), (14) and (15) are modified Bessel functions [9].

A particular solution of (13) is

$$\Phi^*(r, v, p, s) = 2\pi K_{iv/1-q}\left(\frac{f_1 r_0}{1-q}\right) I_{iv/1-q}\left(\frac{f_1 r}{1-q}\right) \qquad (r < r_0) \tag{19}$$

$$\Phi^*(r, v, p, s) = 2\pi K_{iv/1-q}\left(\frac{f_1 r}{1-q}\right) I_{iv/1-q}\left(\frac{f_1 r_0}{1-q}\right) \qquad (r > r_0). \tag{20}$$

While the complementary solution is

$$\Phi^*(r, v, p, s) = C_1(v) K_{iv/1-q}\left(\frac{f_1 r}{1-q}\right)$$

where $I_{iv/1-q}$ and $K_{iv/1-q}$ are modified Bessel functions of imaginary order $iv/1-q$ of the first and second kind respectively, and $C_1(v)$ is a constant coefficient to be determined by the boundary conditions.

Similarly solutions of (14) and (15) are

$$G^*(r, v, p, s) = C_2(v) K_{iv/1-q}\left(\frac{f_2 r}{1-q}\right) \qquad (r \geqslant a) \tag{21}$$

$$H^*(r, v, p, s) = C_3(v) K_{iv/1-q}\left(\frac{f_2 r}{1-q}\right) \qquad (r \geqslant a) \tag{22}$$

where C_2 and C_3 are constants.

The constants C_1, C_2 and C_3 can be expressed as

$$C_1 = \frac{C_{11}(v)}{\Delta(v)}, \quad C_2 = \frac{C_{22}(v)}{\Delta(v)}, \quad C_3(v) = \frac{C_{33}(v)}{\Delta(v)}$$

where

$$\begin{aligned}\Delta(v) = {} & \frac{v^2}{a^2}(f_2^2 + p^2 s^2) K_{iv/1-q}\left(\frac{f_1 a}{1-q}\right) K_{iv/1-q}\left(\frac{f_2 a}{1-q}\right) K_{iv/1-q}\left(\frac{f_2 a}{1-q}\right) \\ & + \frac{f_2^2}{(1-q)^2} K'_{iv/1-q}\left(\frac{f_2 a}{1-q}\right)\left[f_1 f_2 K'_{iv/1-q}\left(\frac{f_1 a}{1-q}\right) K_{iv/1-q}\left(\frac{f_2 a}{1-q}\right)\right. \\ & \left. + p^2 s^2 K_{iv/1-q}\left(\frac{f_1 a}{1-q}\right) K'_{iv/1-q}\left(\frac{f_2 a}{1-q}\right)\right]\end{aligned} \tag{23}$$

$$C_{11}(v) = 2\pi K_{iv/1-q}\left(\frac{f_1 r_0}{1-q}\right)\left\{-\frac{v^2}{a^2}(f_2^2 + p^2 s^2) I_{iv/1-q}\left(\frac{f_1 a}{1-q}\right)\right.$$

$$\times K_{iv/1-q}\left(\frac{f_2 a}{1-q}\right)K_{iv/1-q}\left(\frac{f_2 a}{1-q}\right)-\frac{f_2^2}{(1-q^2)}K'_{iv/1-q}\left(\frac{f_2 a}{1-q}\right)$$

$$\times\left[f_1 f_2 K_{iv/1-q}\left(\frac{f_2 a}{1-q}\right)I'_{iv/1-q}\left(\frac{f_1 a}{1-q}\right)+p^2 s^2 I_{iv/1-q}\right.$$

$$\left.\left.\times\left(\frac{f_1 a}{1-q}\right)K'_{iv/1-q}\left(\frac{f_2 a}{1-q}\right)\right]\right\} \tag{24}$$

$$C_{22}(v)=-\frac{2\pi v^2 s^2 \beta^{-2}}{a^3}K_{iv/1-q}\left(\frac{f_1 r_0}{1-q}\right)K_{iv/1-q}\left(\frac{f_2 a}{1-q}\right) \tag{25}$$

$$C_{33}(v)=\frac{2\pi p^2 s^2 f_2}{a(1-q)}K_{iv/1-q}\left(\frac{f_1 r_0}{1-q}\right)K'_{iv/1-q}\left(\frac{f_2 a}{1-q}\right). \tag{26}$$

Finally the exact solutions are

$$U_r(r,\theta,z,t)=T\int_{L_1}^{U_1}\int_{L_2}^{U_2}\int_{L_3}^{U_3}\left(\frac{\partial\Phi^*}{\partial r}+\frac{v}{r}G^*+ps\frac{\partial H^*}{\partial r}\right)$$
$$\times\exp(v\theta+psz+st)\mathrm{d}v\,\mathrm{d}(ps)\mathrm{d}s \tag{27}$$

$$U_\theta(r,\theta,z,t)=T\int_{L_1}^{U_1}\int_{L_2}^{U_2}\int_{L_3}^{U_3}\left(\frac{v}{r}\Phi^*-\frac{\partial G^*}{\partial r}+vps\frac{H^*}{r}\right)$$
$$\times\exp(v\theta+psz+st)\mathrm{d}v\,\mathrm{d}(ps)\mathrm{d}s \tag{28}$$

$$U_z(r,\theta,z,t)=T\int_{L_1}^{U_1}\int_{L_2}^{U_2}\int_{L_3}^{U_3}\left[sp\Phi^*-\left(\frac{\partial^2 H^*}{\partial r^2}+\frac{1}{r}\frac{\partial H^*}{\partial r}+\frac{v^2}{r^2}H^*\right)\right.$$
$$\times\exp(v\theta+psz+st)\mathrm{d}v\,\mathrm{d}(ps)\mathrm{d}s \tag{29}$$

with

$$\Phi^*=2\pi K_{iv/1-q}\left(\frac{f_1 r_0}{1-q}\right)I_{iv/1-q}\left(\frac{f_1 r}{1-q}\right)+\frac{C_{11}(v)}{\Delta(v)}K_{iv/1-q}\left(\frac{f_1 r}{1-q}\right) \tag{30}$$

$$G^*=\frac{C_{22}(v)}{\Delta(v)}K_{iv/1-q}\left(\frac{f_2 r}{1-q}\right) \tag{31}$$

$$H^*=\frac{C_{33}(v)}{\Delta(v)}K_{iv/1-q}\left(\frac{f_2 r}{1-q}\right) \tag{32}$$

$$T=i/8\pi^3$$

$$L_1=\gamma_1-i\infty,\quad L_2=\gamma_2-i\infty,\quad L_3=\gamma_3-i\infty,$$

$$U_1=\gamma_1+i\infty,\quad U_2=\gamma_2+i\infty,\quad U_3=\gamma_3+i\infty.$$

3. Evaluation of integrals

Now we proceed to evaluate the integrals (27), (28) and (29) by using Residue Cagniard method [2] for the observational point in the shadow region as shown in figure 2.

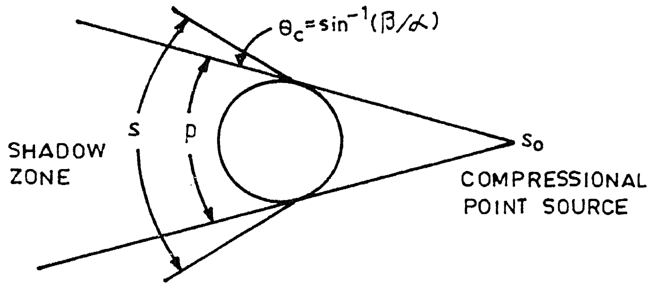

Figure 2. The Projection of the $z = 0$ plane.

For this we have assumed s, the Laplace transform variable to be large, real and positive. We know that this corresponds to the short-time approximations of pulses and diffraction of short-harmonic waves [3].

For an early time response corresponding to the high frequency approximation, let the radial part of the displacement be

$$\hat{U}_r(r,\theta,p,s) = \frac{1}{2\pi i}\int_{L_3}^{U_3}\left[\frac{2\pi f_1}{1-q}K_{i\nu/1-q}\left(\frac{f_1 r_0}{1-q}\right)I'_{i\nu/1-q}\left(\frac{f_1 r}{1-q}\right)\right.$$
$$+\frac{1}{\Delta}\left(\frac{C_{11}f_1}{1-q}\right)K'_{i\nu/1-q}\left(\frac{f_1 r}{1-q}\right) + \frac{C_{22}}{r}K_{i\nu/1-q}\left(\frac{f_1 r}{1-q}\right)$$
$$\left. + \frac{psf_2C_{33}}{1-q}K'_{i\nu/1-q}\left(\frac{f_2 r}{1-q}\right)\right]\exp(\nu\theta)\mathrm{d}\nu. \tag{33}$$

For the region of $\nu \sim f_1 a$, $\nu > f_1 a$ as s becomes large, the asymptotic expansion of the modified Bessel function can be expressed as

$$K_{i\nu}(\chi) \sim [2^{1/2}\pi/(3\chi^{1/3})]\exp(-\nu\pi/2)f(\zeta) \tag{34}$$

$$I_{i\nu}(\chi) \sim [2^{1/2}/(3\chi^{1/3})]\exp(\pi\nu/2)[if(\zeta)+g(\zeta)] \tag{35}$$

$$K'_{i\nu}(\chi) \sim [-2^{1/2}\pi/(3\chi^{2/3})]\exp(-\pi\nu/2)f'(\zeta) \tag{36}$$

$$I'_{i\nu}(\chi) \sim [2^{1/2}/(3\chi^{2/3})]\exp(\pi\nu/2)[if'(\zeta)+g'(\zeta)] \tag{37}$$

where

$$f(\zeta) = 3(2)^{-1/6}A_i(-2^{1/3}\zeta) \tag{38}$$

$$g(\zeta) = 3(2)^{1/6}B_i(-2^{1/3}\zeta) \tag{39}$$

A_i and B_i are Airy functions, and ζ is defined as

$$\zeta = 2^{-1}[3\chi(\xi\cosh\xi - \sinh\xi)]^{2/3}\cos\xi = \nu/\chi. \tag{40}$$

Using the above expressions, we write (23) in the form

$$\Delta(\nu) \sim 2^{1/2}\pi 3^{-1}a^{-2}\nu^2\left(\frac{f_1 a}{1-q}\right)^{-1/3}s^2\beta_0^{-2}r^{-2q}K_{i\nu/1-q}\left(\frac{f_2 a}{1-q}\right)K_{i\nu/1-q}$$
$$\times\left(\frac{f_2 a}{1-q}\right)f(\zeta)\exp(-\pi\nu/2(1-q)). \tag{41}$$

Thus the zeros of $\Delta(v)$ for large s correspond to those of $f(\zeta)$, we denote the latter by ζ_n, which can be related to v_n as

$$v_n \sim f_1 a + \zeta_n (f_1 a)^{1/3}, \quad n = 1, 2, 3 \ldots \tag{42}$$

The contour in (33) is chosen to be closed in the right half plane for $\theta < 0$ and in the left half plane for $\theta > 0$. Here the Jordan's Lemma is applicable. Hence there are no contributions to the integrals of (33) other than the ones arising from the poles. Then the series of residues of (33) can be written as

$$\hat{U}_r(r, \theta, p, s) =$$

$$\sum_{n=1}^{\infty} \frac{\dfrac{C_{11} f_1}{1-q} K'_{iv_n/1-q}\left(\dfrac{f_1 r}{1-q}\right) + \dfrac{v_n C_{22}}{r} K_{iv_n/1-q}\left(\dfrac{f_2 r}{1-q}\right) + \dfrac{ps f_2 C_{33}}{1-q} K'_{iv_n/1-q}\left(\dfrac{f_2 r}{1-q}\right)}{\left(\dfrac{\partial \Delta}{\partial v}\right) v = v_n}$$

$$\times \exp(|\theta| v_n) \tag{43}$$

The motion can be divided into compressional and shear motions. We consider the P-wave, which corresponds to the first term of (43) and denote it as $\bar{U}_{r,\mathrm{PP_dP}}(r, \theta, p, s)$. For further evaluation we have

$$\bar{U}_{r,\mathrm{PP_dP}}(r, \theta, z, s) = \frac{1}{2\pi i} \int_{L_2}^{U_2} \hat{U}_{r,\mathrm{PP_dP}}(r, \theta, p, s) \exp(psz) \mathrm{d}(ps). \tag{44}$$

Because of the complexity of evaluation of (44) we use a simple formula for the Bessel functions which are ([3])

$$K_{iv}(\chi) \sim (\pi/2)^{1/2} (\chi^2 - v^2)^{-1/4} \exp\left\{ v \cos^{-1}\frac{v}{\chi} - (\chi^2 - v^2)^{1/2} - \frac{v\pi}{2} \right\} \tag{45}$$

$$K'_{iv}(\chi) \sim (\pi/2) (\chi^2 - v^2)^{1/4} \chi^{-1} \exp\left\{ v \cos^{-1}\frac{v}{\chi} - (\chi^2 - v^2)^{1/2} - \frac{v\pi}{2} \right\}. \tag{46}$$

Substituting the above expressions of (45) and (46) into (44) we get

$$\bar{U}_{r,\mathrm{PP_dP}}(r, \theta, z, s) = \sum_{n=1}^{\infty} \int_{L_2}^{U_2} A_1 F_{n,p} s^{4/3} M^{1/3}$$

$$\times \exp[-s(MD_p - zp) - s^{1/3} M^{1/3} E_{n,p}] \mathrm{d}p \tag{47}$$

where

$$A_1 = i 2^2 a^{1/3} r^{-1} (r_0^2 - a^2)^{-1/4} (r^2 - a^2)^{1/4} \tag{48}$$

$$F_{n,p} = g(\zeta_n)/f'(\zeta_n) \tag{49}$$

$$D_p = \frac{a}{(1-q)} [(r^2 a^{-2} - 1)^{1/2} + (r_0^2 a^{-2} - 1)^{1/2} + \Delta_p] \tag{50}$$

$$\Delta_p = (1-q)|\theta| - \cos^{-1}\frac{a}{r} - \cos^{-1}\frac{a}{r_0} \tag{51}$$

$$E_{n,p} = \frac{a^{1/3}}{(1-q)} \Delta_p \zeta_n, \quad M = \left(\frac{1}{\alpha_0^2 r^{2q}} - p^2 \right)^{1/2}. \tag{52}$$

In (51), Δ_p, the angle in the shadow zone measured from the shadow zone boundary must be positive such that Jordan's Lemma can be applied and (47) is valid in the region of the geometric shadow.

Analysing (47), we find that there are four branch points namely $p = \pm \alpha_0^{-1} r^{-q}$ and $p = \pm \beta_0^{-1} r^{-q}$. We choose the branch cuts along the real axis given by

$$\frac{1}{\alpha_0 r^q} < |\mathrm{real}\,(p)| < \infty \quad \text{and} \quad \frac{1}{\beta_0 r^q} < |\mathrm{real}\,(p)| < \infty$$

Change the variables as

$$\tau_p = MD_p - zp. \tag{53}$$

So that path of integration is modified such that Γ_p, which can be represented by a hyperbola in the p-plane as

$$p = -\frac{z}{L_p^2}\tau_p \pm i \frac{D_p}{L_p^2}\left(\tau_p^2 - \frac{L_p^2}{\alpha_0^2 r^{2q}} \right)^{1/2} \tag{54}$$

with

$$L_p^2 = D_p^2 + z^2. \tag{55}$$

The point of intersection of the hyperbola with the real axis in the p plane is between the branch point $-1/\alpha_0 r^q$ and the origin (figure 3).

Now (47) becomes

$$\bar{U}_{r,\mathrm{PP_dP}}(r,\theta,z,s) = A_1 \sum_{n=1}^{\infty} \int_{\tau_1}^{\infty} F_{n,p} s^{4/3} M^{1/3} G_p \times \exp(-s\tau_p - s^{1/3} M^{1/3} E_{n,p}) \mathrm{d}\tau_p \tag{56}$$

$$G_p = \frac{\mathrm{d}p}{\mathrm{d}\tau_p} \tag{57}$$

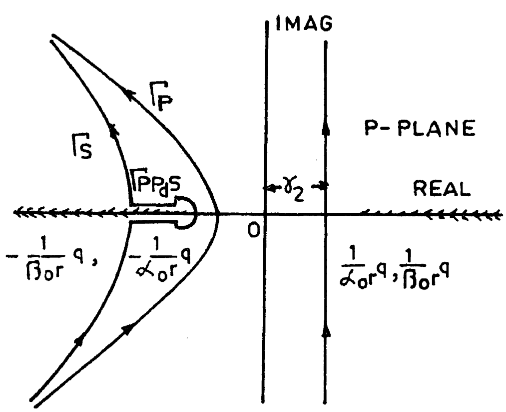

Figure 3. The paths of integration of the P wave (Γ_P), S wave (Γ_S) and $\mathrm{PP_dS}$ wave ($\Gamma_{\mathrm{PP_dS}}$).

τ_1 is the first arrival time of the diffracted PP_dP waves. Then the result for the calculation of the inverse Laplace transform of the P-wave motion is obtained as

$$U_{r,PP_dP}(r,\theta,z,t)=\frac{A_1}{2\pi i}\sum_{n=1}^{\infty}\text{Imag}\int_{\tau_1}^{\infty}\int_{L_1}^{U_1}F_{n,p}s^{4/3}M^{1/3}G_p \times\exp[s(t-\tau_p)-s^{1/3}M^{1/3}E_{n,p}]\,d\tau_p\,ds, \tag{58}$$

where Imag $\int$ is the imaginary part of the integration. Equation (58) can be evaluated by means of the following formula (Ragab [7]).

$$\frac{1}{2\pi i}\int_{L_1}^{U_1}s^m\exp(sT-s^{1/3}Q)\,ds\sim(2\pi^{1/2}\ 3(6m+1)/4)^{-1}\frac{Q^{(6m+3)/4}}{T^{(6m+3)/4}} \times\exp\left(\frac{-2Q^{3/2}}{3^{3/2}T^{1/2}}\right)H(T). \tag{59}$$

Here we have

$$m=\frac{4}{3},\quad T=t-\tau_p,\quad Q=M^{1/3}E_{n,p}H(T)=\text{the Heaviside function.}$$

Finally the diffracted PP_dP wave can be obtained in the domain as

$$U_{r,PP_dP}(r,\theta,z,t)=A_2\sum_{n=1}^{\infty}F_{n,p}E_{n,p}^{11/4}\,\text{Imag}\int_{\tau_1}^{t}\frac{M^{5/4}G_p}{(t-\tau_p)^{13/4}}\exp\left(\frac{-2M^{1/2}E_{n,p}^{3/2}}{3^{3/2}(t-\tau_p)^{1/2}}\right) \times H(t-\tau_p)\,d\tau_p \tag{60}$$

where

$$A_2=-i2^{-3/2}\pi^{-1/2}3^{-9/4}a^{1/3}r^{-1}(r_0^2-a^2)^{-1/4}(r^2-a^2)^{1/4}.$$

Now, we propose to obtain the diffracted S waves which correspond to the second term (M-motion) and the third term (N-motion) of (43).

If we follow the technique used above we find, after a little calculation, that these diffracted events are obtained as follows:

$$U_{r,M,PS_dS}(r,\theta,z,t)=\sum_{n=1}^{\infty}A_m\,\text{Imag}\int_{\tau_2}^{t}\frac{M^{5/2}G_sE_{s,n}^4}{(t-\tau_s)^{9/2}} \times\exp\left(\frac{-2M^{1/2}E_{s,n}^{3/2}}{3^{3/2}(t-\tau_s)^{1/2}}\right)H(t-\tau_s)\,d\tau_s \tag{61}$$

$$U_{r,N,PS_dS}(r,\theta,z,t)=\sum_{n=1}^{\infty}A_n\,\text{Imag}\int_{\tau_2}^{t}\frac{Q_1N^3E_{s,n}^{11/2}G_s}{(t-\tau_s)^6}\exp\left(\frac{-2M^{1/2}E_{s,n}^{3/2}}{3^{3/2}(t-\tau_s)^{1/2}}\right) \times H(t-\tau_s)\,d\tau_s, \tag{62}$$

where the path of integration for these motions is along Γ_s (figure 3). Similarly the diffracted PP_dS waves are obtained as

$$U_{r,M,PP_dS}(r,\theta,z,t)=\sum_{n=1}^{\infty}A_m\,\text{Imag}\int_{\tau_3}^{t}\frac{M^{5/2}G_{ps}E_{s,n}^4}{(t-\tau_{ps})^{9/2}}\exp\left(\frac{-2M^{1/2}E_{s,n}^{3/2}}{3^{3/2}(t-\tau_{ps})^{1/2}}\right) \times H(t-\tau_{ps})\,d\tau_{ps} \tag{63}$$

$$U_{r,N,\mathrm{PS_dS}}(r,\theta,z,t)=\sum_{n=1}^{\infty}A_n\,\mathrm{Imag}\int_{\tau_3}^{t}\frac{Q_1{}^{N3}E_{s,n}^{11/2}G_{ps}}{(t-\tau_{ps})^6}\exp\left(\frac{-2M^{1/2}E_{s,n}^{3/2}}{3^{3/2}(t-\tau_{ps})^{1/2}}\right)\times H(t-\tau_{ps})\mathrm{d}\tau_{ps} \tag{64}$$

where the path of integrations is along $\Gamma_{\mathrm{PP_dS}}$ (figure 3),

$$A_m=i2^{-2}3^{-5/2}\pi^{-1}a^{2/3}(1-q)^{-5/6}[rf'(\zeta_n)]^{-1}(r^2-a^2d^2)^{-1/4}\times(r_0^2-a^2)^{-1/4}(1-d^2)^{1/4} \tag{65}$$

$$A_n=i2^{-2}3^{-4}\pi^{-1}a^{7/6}(1-q)^{-11/6}(\beta_0 r^q)^2[f'(\zeta_n)]^{-1}(r_0^2-a^2)^{-1/4}\times(r^2-a^2d^2)^{-1/4}(1-d^2)^{3/4} \tag{66}$$

$Q_1=p^3$, τ_2 and $\tau_3=$ the first arrival time of the $\mathrm{PS_dS}$ and $\mathrm{PP_dS}$ respectively.

$$E_{s,n}=\frac{a^{1/3}\Delta_s\zeta_n}{(1-q)},\quad \Delta_s=|\theta|(1-q)-\cos^{-1}\frac{a}{r_0}-\cos^{-1}\frac{ad}{r}+\cos^{-1}(\mathrm{d}) \tag{67}$$

$$d=\beta_0/\alpha_0 \tag{68}$$

$$p=-(z/L_s^2)\tau_{s-}+i(I+J)L_s^{-2}\sqrt{\tau_s^2-w_0^2L_s^2},\quad G_s=\frac{\mathrm{d}p}{\mathrm{d}\tau_s} \tag{69}$$

$$L_s^2=z^2+(I+J)^2,\quad \tau_s=ND_s-zp \tag{70}$$

$$W_0=(I^2\alpha_0^2r^{-2q}+J^2\beta_0^{-2}r^{-2q})^{1/2}(I+J)^{-1} \tag{71}$$

$$I=\frac{a\Delta_s+(r_0^2-a^2)^{1/2}}{(1-q)},\quad J=\frac{(r^2-a^2d^2)^{1/2}-a(1-d^2)}{(1-q)}. \tag{72}$$

For the path of integration $\Gamma_{\mathrm{PP_dS}}$ (figure 3)

$$p=z\tau_{ps}(L_s)^{-2}\pm(I+J)W_2(L_s)^{-2}\pm i\delta,\quad G_{ps}=\frac{\mathrm{d}p}{\mathrm{d}\tau_{ps}} \tag{73}$$

with

$$W_2=(L_s^2W_0^2-\tau_{ps}^2)^{1/2}. \tag{74}$$

4. Numerical results and discussion

The numerical results as shown here are for the case in which all quantities are normalised with respect to the radius a of the cylinder and the wave velocity α_0. For convenience the material properties λ, μ and ρ are chosen such that $\alpha_0/\beta_0=3$ and the Poisson's ratio is assumed to be 1/4.

Figures 4, 5 and 6 show the variations of the radial displacement of the diffracted $\mathrm{PP_dP}$ wave, the diffracted M-type $\mathrm{PS_dS}$ wave and the diffracted N-type $\mathrm{PS_dS}$ waves, a function of time. Four cases are plotted with the source position $r_0=3a$, $z_0=0$, $\theta_0=0$ and the observational positions $r=2a$, $z=a$, $\theta=180°$, 170°, 160° and 150°. The figures also show that the magnitude of the diffracted $\mathrm{PP_dP}$ waves changes more rapidly than that of the diffracted M and N type $\mathrm{PS_dS}$ waves, when θ changes from 150° to 180°. However, our result shows that $\mathrm{PP_dP}$ waves do not exist when $\theta<150°$.

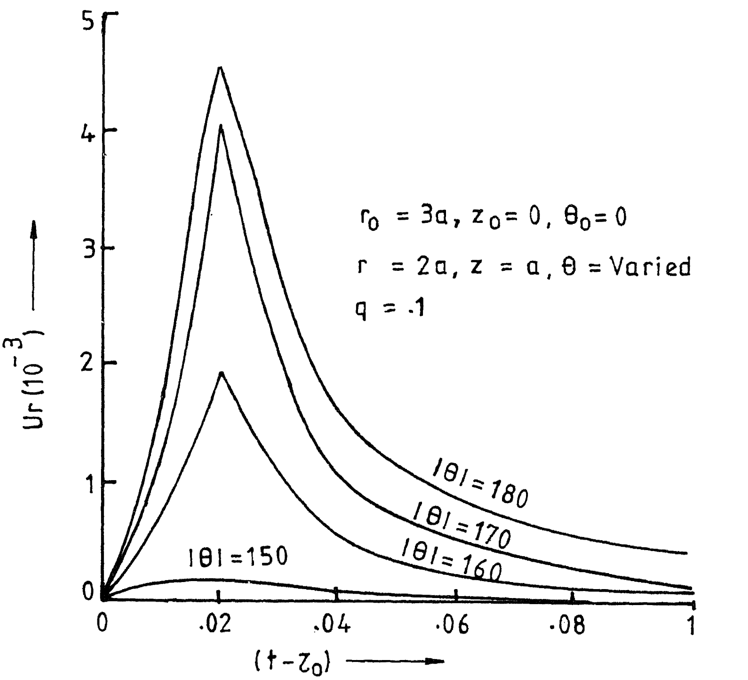

Figure 4. Variation of the radial displacement of the diffracted PP_dP wave as a function of time.

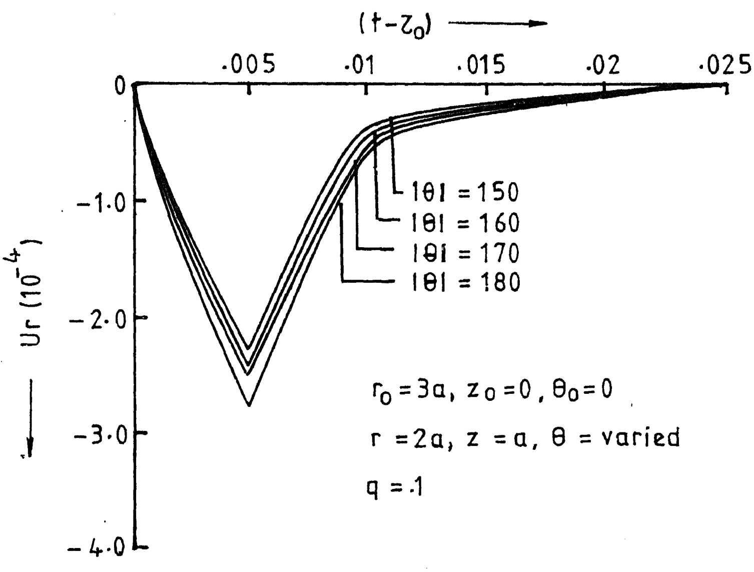

Figure 5. Variation of the radial displacement of the diffracted *M*-type PS_dS wave as a function of time.

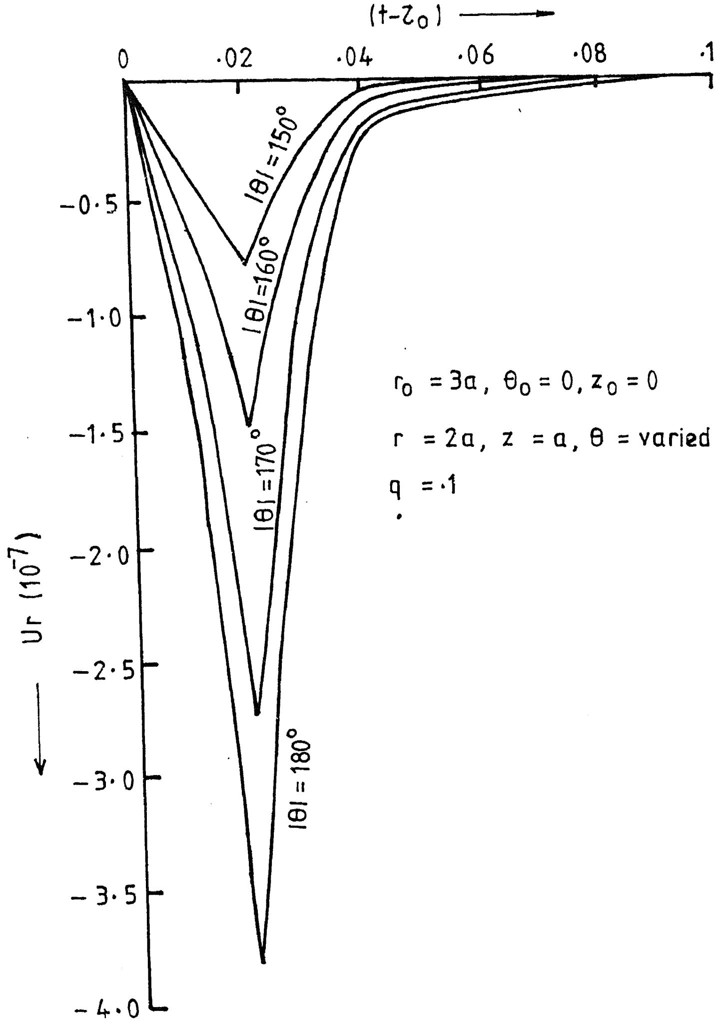

Figure 6. Variation of the radial displacement of the diffracted N-type PS_dS wave as a function of time.

The short time approximations for the diffracted pulses by a rigid cylinder in a homogeneous medium can be obtained by putting $q=0$ in our result. The results obtained agree with those obtained by Hwang *et al* [4].

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Topological algebras with C^*-enveloping algebras

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Abstract. Let A be a complete topological *algebra which is an inverse limit of Banach *algebras. The (unique) enveloping algebra $\mathscr{E}(A)$ of A, providing a solution of the universal problem for continuous representations of A into bounded Hilbert space operators, is known to be an inverse limit of C^*-algebras. It is shown that $\mathscr{E}(A)$ is a C^*-algebra iff A admits greatest continuous C^*-seminorm iff the continuous states (respectively, continuous extreme states) constitute an equicontinuous set. A Q-algebra (i.e., one whose quasiregular elements form an open set) A has C^*-enveloping algebra. There exists (i) a Frechet algebra with C^*-enveloping algebra that is not a Q-algebra under any topology and (ii) a non-Q spectrally bounded algebra with C^*-enveloping algebra. A hermitian algebra with C^*-enveloping algebra turns out to be a Q-algebra. The property of having C^*-enveloping algebra is preserved by projective tensor products and completed quotients, but not by taking closed subalgebras. Several examples of topological algebras with C^*-enveloping algebras are discussed. These include several pointwise algebras of functions including well-known test function spaces of distribution theory, abstract Segal algebras and concrete convolution algebras of harmonic analysis, certain algebras of analytic functions (with Hadamard product) and Köthe sequence algebras of infinite type. The enveloping C^*-algebra of a hermitian topological algebra with an orthogonal basis is isomorphic to the C^*-algebra c_0 of all null sequences.

Keywords. C^*-enveloping algebra; Q-algebra; hermitian algebra; Segal algebra; Köthe sequence space.

1. Introduction

A complete locally m-convex *algebra A is a topological *algebra that is an inverse limit of Banach* algebras. In representation theory of such algebras, the enveloping algebra $(\mathscr{E}(A), \tau)$ of A has been introduced in [10], [21], [16] which provides a solution of the universal problem for continuous *representations of A into bounded Hilbert space operators. This corresponds to the construction of the enveloping C^*-algebra of a Banach* algebra [13, § 2.7.2, p. 48]. The algebra $(\mathscr{E}(A), \tau)$ is a pro-C^*-algebra [27] in the sense that it is an inverse limit of C^*-algebras. This paper is concerned with those A for which $(\mathscr{E}(A), \tau)$ is a C^*-algebra. In fact, in [16], [17], A is called a *bQ-algebra* if $(\mathscr{E}(A), \tau)$ is a barreled space that is a Q-algebra (a topological algebra A is a *Q-algebra* [26] if the set A_{-1} of all quasiregular elements of A is an open set). The barreled assumption turned out to be redundant; for a pro-C^*-algebra, which is a Q-algebra, is a C^*-algebra [18, Corollary 2.2], [27, Proposition 1.14].

Topological *algebras with C^*-enveloping algebras are important for a couple of reasons. Though non-normed, they are well-behaved. In the literature, bQ-condition

has been assumed in several aspects like tensor products [17], hermitian K-theory [24] and representation theory [16]. In fact, the representation theory of such algebras is quite similar to that of Banach* algebras. Further, as exhibited in the present paper, there are several classes of examples of such algebras arising in function theory, Fourier series, abstract harmonic analysis, complex analysis and nuclear spaces, in particular, sequence spaces. In what follows, we briefly describe the contents of the present paper.

In [17], the question of completely specifying the class of bQ-algebras was discussed. We show that a complete lmc-*algebra A has C^*-enveloping algebra (i.e., A is a bQ-algebra) iff A admits greatest continuous C^*-seminorm $p_\infty(\cdot)$ iff the continuous states (respectively, continuous extreme states) constitute an equicontinuous set. This is used to show that the enveloping algebra of a Q lmc-*algebra is a C^*-algebra, but the converse does not hold. An lmc-*algebra A is spectrally bounded (sb) (respectively, *spectrally bounded (*sb)) if the spectrum of each $x \in A$ (respectively, the spectrum of each element of the form x^*x) is bounded. We discuss the examples exhibiting: (i) a Frechet algebra with C^*-enveloping algebra, which is not sb, and which fails to be a Q-algebra under any topology and (ii) a non-Q sb algebra with C^*-enveloping algebra. However, if A is hermitian and having C^*-enveloping algebra, then A is a Q-algebra. Further, it is also shown that if A is *sb, then A admits greatest C^*-seminorm $|\cdot|$ (not necessarily continuous); and such an A is hermitian iff $|\cdot| = s(\cdot)$ iff $s(\cdot)$ is a C^*-seminorm. Here $s(x) = r(x^*x)^{1/2}$ $(x \in A)$, $r(\cdot)$ denoting the spectral radius. Thus if A is *sb, then A is hermitian and has C^*-enveloping algebra iff $s(\cdot)$ is a continuous C^*-seminorm (in which case, $s(\cdot) = p_\infty(\cdot)$). We also show that if A is Frechet, then (i) A is sb iff A is a Q-algebra and (ii) if A is *sb, then A has C^*-enveloping algebra. Projective tensor products and complete quotients of algebras with C^*-enveloping algebras are algebras with C^*-enveloping algebras; but the enveloping algebra of a closed *subalgebra of an algebra with C^*-enveloping algebra need not be a C^*-algebra. We have also discussed several classes of algebras with C^*-enveloping algebras. Notable among these, besides pointwise algebras of functions (including the algebra $C^\infty(X)$ of smooth functions on a compact manifold) are the various test function spaces of distribution theory, topological Segal algebras [11] of harmonic analysis (in particular, certain convolution group algebras of locally compact groups) and Köthe G_∞-sequence algebras [22] (of significance in the theory of nuclear and Schwartz spaces). This also incorporates certain topological algebras with orthogonal bases [15], [20]; and via Fourier expansion and Taylor expansion, algebras of smooth periodic functions (convolution product) and of analytic functions (Hadamard product). The enveloping algebra, of an lmc-*algebra with hermitian orthogonal basis and having C^*-enveloping algebra, is *isomorphic to the C^*-algebra c_0 of all null scalar sequences. Let us note that the class of topological *algebras with C^*-enveloping algebras also include the Frechet algebra of C^∞-elements of automorphic action of Lie group on a C^*-algebra, certain Ψ^*-algebras of pseudo-differential operators and the algebra of local observables of quantum field theory. These will be discussed in a subsequent paper.

Preliminaries and notations

A locally m-convex *algebra (lmc-*algebra) [25], [26], [9], [10] is a linear associative involutive algebra A with complex scalars and with a Hausdorff locally convex

topology t on it which is determined by a separating directed family $P=(p_\alpha:\alpha\in\Delta)$ of seminorms satisfying, for all α and for all x,y; $p_\alpha(xy)\leqslant p_\alpha(x)p_\alpha(y)$ (submultiplicativity) and $p_\alpha(x^*)=p_\alpha(x)$ (*-invariance). Let (e_γ) be a bounded approximate identity (bai) for A, i.e. $(e_\gamma)\subset A$ is a net such that (a) for each $x\in A$, $e_\gamma x\to x$, $xe_\gamma\to x$ and (b) for each α, $p_\alpha(e_\gamma)\leqslant 1$ for all γ. One can take P to be the collection $K(A)$ of all continuous *invariant submultiplicative seminorms p satisfying $p(e_\gamma)\leqslant 1$ for all γ. A pro-C^*-algebra [27], [28], [6], [7] (also called an l.m.c. C^*-algebra [16] or a locally C^*-algebra [21]) is a complete lmc-*algebra A in which each p_α is a C^*-seminorm, i.e., each p_α additionally satisfies $p_\alpha(x^*x)=p_\alpha(x)^2$ for all $x\in A$. Given an lmc-*algebra A and $p\in K(A)$, let $N_p=\{x\in A:p(x)=0\}$, and A_p be the Banach *algebra obtained by completing the quotient *algebra A/N_p in the norm $\|x_p\|_p=p(x)$, $x_p=x+N_p$. For a $p_\alpha\in P$, let $A_\alpha=A_{p_\alpha}$. If A is complete, then $A=\varprojlim_{\alpha\in\Delta}A_\alpha=\varprojlim_{p\in K(A)}A_p$, an inverse limit of Banach* algebras [26, Theorem 5.1]. Similarly, a pro-C^*-algebra is an inverse limit of C^*-algebras. An lmc-*algebra A is hermitian if for each $h=h^*$ in A, the spectrum $\mathrm{sp}(h)\subset\mathbb{R}$.

Let A be a complete lmc-*algebra with a bai (e_γ). In representation theory of such algebras [10], [21], [18], the enveloping algebra $(\mathscr{E}(A),\tau)$ of a has been introduced as follows. Let $R(A)$ (respectively, $R'(A)$) be the set of all continuous (respectively, continuous topologically irreducible) *representations $\pi:A\to B(H_\pi)$ of A into the C^*-algebras $B(H_\pi)$ of all bounded linear operators on Hilbert spaces H_π. For a $p\in K(A)$, let $R_p(A)=\{\pi\in R(A)$: there exists $k>0$ such that $\|\pi(x)\|\leqslant kp(x)$ for all $x\}$, $R'_p(A)=R_p(A)\cap R'(A)$, $R_\alpha(A)=R_{p_\alpha}(A)$, $R'_\alpha(A)=R'_{p_\alpha}(A)$. Then $R(A)=\cup_\alpha R_\alpha(A)=\cup\{R_p(A):p\in K(A)\}$, $R'(A)=\cup_\alpha R'_\alpha(A)=\cup\{R'_p(A):p\in K(A)\}$. For $p\in K(A)$, $r_p(x)=\sup\{\|\pi(x)\|:\pi\in R_p(A)\}=\sup\{\|\pi(x)\|:\pi\in R'_p(A)\}$ [16, Lemma 4.1] $(x\in A)$ defines a continuous C^*-seminorm on A. Let $r_\alpha(\cdot)=r_{p_\alpha}(\cdot)$. The *radical of A is the *ideal $s\mathrm{rad}\,A=\cap_\alpha N(r_\alpha)=\cap\{N(r_p):p\in K(A)\}$, where $N(r_p)=\{x\in A:r_p(x)=0\}$. The algebra $(\mathscr{E}(A),\tau)$ is the Hausdorff completion of $(A,\{r_\alpha\})$ (equivalently, of $(A,\{r_p:p\in K(A)\})$), i.e. the completion of $A/s\mathrm{rad}\,A$ in the topology τ defined by C^*-seminorms $q_\alpha(x+s\mathrm{rad}\,A)=\inf\{r_\alpha(x+i):i\in s\mathrm{rad}\,A\}=r_\alpha(x)\,(x\in A)$. The pro-$C^*$-algebra $(\mathscr{E}(A),\tau)=\varprojlim\mathscr{E}(A_p)=\varprojlim\varepsilon(A_\alpha)$, where $\mathscr{E}(A_p)$ is the enveloping C^*-algebra of the Banach *algebra A_p [13, §2.7.2, p. 48]. Let $\phi:A\to\mathscr{E}(A)$ be $\phi(x)=x+s\mathrm{rad}\,A$. The algebra $(\mathscr{E}(A),\tau)$ satisfies the universal property that, given $\pi\in R(A)$ (respectively, $\pi\in R'(A)$) there exists a unique $\sigma\in R(\mathscr{E}(A))$ (respectively, $\sigma\in R'(\mathscr{E}(A))$) such that $\pi=\sigma\circ\phi$ [16, p. 69–70]. Further, it is easily seen that $(\mathscr{E}(A),\tau)$ is a unique (up to a homeomorphic *isomorphism) pro-C^*-algebra satisfying this universal property. Thus the following unambiguously makes sense.

DEFINITION

A complete lmc-*algebra A has C^*-enveloping algebra if $(\mathscr{E}(A),\tau)$ is a C^*-algebra.

2. Basic theory of algebras with C^*-enveloping algebras

Throughout the section, A denotes a complete lmc-*algebra with a bai. The following corresponds to the fact that a Banach* algebra admits greatest C^*-seminorm (automatically continuous) viz the Gelfand-Naimark pseudonorm [8, §39].

Theorem 2.1. *The algebra A has C^*-enveloping algebra iff A admits greatest continuous C^*-seminorm. In this case, if $p_\infty(\cdot)$ dentoes the greatest continuous C^*-seminorm on A,*

then $p_\infty(\cdot) = sup_\alpha r_\alpha(x) = sup\{\|\pi(x)\|:\pi\in R(A)\} = sup\{\|\pi(x)\|:\pi\in R'(A)\}$ $(x\in A)$; *and* $(\mathscr{E}(A),\tau)$ *is the* C^**-algebra* $(A/N(p_\infty))\tilde{}$, *the completion of* $A/N(p_\infty)$ *in the norm* $\|x+N(p_\infty)\|_\infty = p_\infty(x)$, $N(p_\infty)=\{x\in A:p_\infty(x)=0\}$.

Proof. Observe that on $A/s\,\mathrm{rad}\,A$, $q_\alpha(x+s\,\mathrm{rad}\,A)=r_\alpha(x)$ for each $\alpha\in\Delta$. Indeed, for any $x\in A$, $q_\alpha(x+s\,\mathrm{rad}\,A)=\inf\{r_\alpha(x+i):i\in s\,\mathrm{rad}\,A\}=\inf_{i\in I}[\sup\{f((x+i)^*(x+i))^{1/2}: f\in P_\alpha(A)\}]$ using $r_\alpha(z)=\sup\{f(z^*z)^{1/2}:f\in P_\alpha(A)\}$ [16, Lemma 4.1], where $P_\alpha(A)$ denotes the set of all continuous positive linear functionals f on A such that $|f(u)|\leqslant p_\alpha(u)$ for all $u\in A$. Since $i\in s\,\mathrm{rad}\,A$, $r_\alpha(i)=0$; and so $f(i^*i)=0$ for all $f\in P_\alpha(A)$. Further, for all such f, by the Cauchy-Schwarz inequality, $f(i^*x)=0=f(x^*i)$ for all $x\in A$. Hence

$$\begin{aligned} q_\alpha(x+s\,\mathrm{rad}\,A) &= \inf[\sup\{f(x^*x)+f(i^*x)+f(x^*i)+f(i^*i):i\in I\}^{1/2}:f\in P_\alpha(A)] \\ &= \sup\{f(x^*x)^{1/2}:f\in P_\alpha(A)\} = r_\alpha(x). \end{aligned}$$

Now suppose that A has C^*-enveloping algebra, so that $(\mathscr{E}(A),\tau)$ is a C^*-algebra, the topology τ being determined by a C^*-norm $\|\cdot\|$. By [27, p. 165], for any $z\in\mathscr{E}(A)$, $\sup_\alpha q_\alpha(z)<\infty$, and $\|z\|=\sup_\alpha q_\alpha(z)$. Thus $p_\infty(x)=\|x+s\,\mathrm{rad}\,A\|=\sup_\alpha r_\alpha(x)$ $(x\in A)$ defines a C^*-seminorm on A; and there exists $k>0$ and $\alpha\in\Delta$ such that for all $x\in A$, $p_\infty(x)=\|x+s\,\mathrm{rad}\,A\|\leqslant kq_\alpha(x+s\,\mathrm{rad}\,A)=kr_\alpha(x)\leqslant kp_\alpha(x)$ using [16, p. 69]. Let p be any continuous C^*-seminorm on A, so that, for some $l>0$ and some $\beta\in\Delta$, $p(x)\leqslant lp_\beta(x)$ $(x\in A)$. Then $R_p(A)\subset R_\beta(A)$ and for all x, $r_p(x)\leqslant r_\beta(x)$. Identifying $R_p(A)$ and $R(A_p)$ canonically [16, Proposition 3.5] and using that A_p is a C^*-algebra; it follows that for each x, $p(x)=\|x+N(p)\|_p=\sup\{\|\pi(x+N(p))\|:\pi\in R(A_p)\}=\sup\{\|\pi(x)\|:\pi\in R_p(A)\}=r_p(x)\leqslant r_\beta(x)\leqslant p_\infty(x)$. Thus $p_\infty(\cdot)$ is the greatest continuous C^*-seminorm on A.

Conversely, let A admit greatest continuous C^*-seminorm, say $p_\infty(\cdot)$. There exist $\beta\in\Delta$, $k>0$ such that for all $x\in A$, $p_\infty(x)\leqslant kp_\beta(x)$. Hence, as above $p_\infty(x)\leqslant r_\beta(x)$ and so $p_\infty(x)=r_\beta(x)$ for all x. Since each $r_\alpha(\cdot)$ satisfies $r_\alpha(x)\leqslant p_\alpha(x)$ $(x\in A)$ [16, p. 69], $r_\alpha(x)\leqslant p_\infty(x)$ for all α, for all x. Thus $p_\infty(x)=\sup_\alpha r_\alpha(x)$ $(x\in A)$, $s\,\mathrm{rad}\,A=N(p_\infty)$, and for any x, $\|x+s\,\mathrm{rad}\,A\|_{p_\infty}=p_\infty(x)=\sup_\alpha q_\alpha(x+s\,\mathrm{rad}\,A)=r_\beta(x)=q_\beta(x+s\,\mathrm{rad}\,A)$. It follows that the topology τ on $\mathscr{E}(A)$ is determined by $\|\cdot\|_{p_\infty}$; and then $(\mathscr{E}(A),\tau)$ is a C^*-algebra. This completes the proof.

COROLLARY 2.2.

If A is a Q-algebra, then A has C^*-enveloping algebra.

Proof. Let A be a Q-algebra. By [26, Lemma E.3] A is sb; and [26, Proposition 13.5] implies that there exists an $\alpha_0\in\Delta$ and $k>0$ such that $r(x)\leqslant kp_{\alpha_0}(x)$ for all x. Let q be any continuous C^*-seminorm on A. There exists $p\in K(A)$ and $M>0$ such that $q(x)\leqslant Mp(x)$ for all x. Then for any $h=h^*$ in A and for $n=1,2,3,\ldots$, $q(h)=q(h^{2^n})^{1/2^n}\leqslant M^{1/2^n}p(h^{2^n})^{1/2^n}$. By the spectral radius formula [26, p. 22], $q(h)\leqslant\lim_{n\to\infty}\sup M^{1/2^n}\cdot p_\alpha(h^{2^n})^{1/2^n}\leqslant\sup_{p\in K(A)}\lim_{n\to\infty}\sup p(h^n)^{1/n}=r(h)\leqslant kp_{\alpha_0}(h)$. Hence for any $x\in A$, $q(x)=q(x^*x)^{1/2}\leqslant k^{1/2}p_{\alpha_0}(x)$. Thus $p_\infty(x)=\sup\{q(x):q$ is a continuous C^*-seminorm on $A\}\leqslant k^{1/2}p_{\alpha_0}(x)(x\in A)$, and $p_\infty(\cdot)$ is the greatest continuous C^*-seminorm.

Let A be commutative. Let $\mathscr{M}(A)$ be the Gelfand space (with weak* topology) of A consisting of all continuous multiplicative linear functionals on A. For $\phi\in\mathscr{M}(A)$,

let $\phi^* \in \mathscr{M}(A)$ be defined as $\phi^*(x) = \overline{\phi(x^*)}$. The hermitian Gelfand space of A is $\mathscr{M}^*(A) = \{\phi \in \mathscr{M}(A): \phi = \phi^*\}$. The following can be shown, as in [8, Theorem 40.2, p. 220] using the machinery in [16].

Lemma 2.3. *Let A be commutative. Let $\pi \in R'(A)$. Then π is one dimensional, and there exists $\phi \in \mathscr{M}^*(A)$ such that $\pi(x) = \phi^*(x)1$ for all $x \in A$.*

Example 2.4. There exists a unital commutative Frechet *algebra B with C^*-enveloping algebra such that B is not sb and B fails to be a Q-algebra (under any topology). Let $U = \{z \in \mathbb{C}: -1 < \mathrm{Re} z < 1\}$. Let $C(\bar{U})$ be the algebra, with pointwise operations, of all continuous complex valued functions on $\bar{U}$ with compact open topology t. Let $B = \{f \in C(\bar{U}): f \text{ is analytic in } U\}$. Then (B, t) is a Frechet *algebra with involution $f \to f^*$, $f^*(z) = \overline{f(\bar{z})}$ $(z \in \bar{U})$. The topology t is defined by the family of seminorms $P = (p_n: n = 0, 1, 2, \ldots)$, where $p_n(f) = \sup\{|f(z)|: z \in K_n\}$, $K_n = \{z \in \bar{U}: n \leqslant \mathrm{Im} z \leqslant n+1\}$. It is easily seen that $\phi \in \mathscr{M}(B)$ iff $\phi = \phi_z$ for some $z \in \bar{U}$, $\phi_z(f) = f(z)$; and $\mathscr{M}^*(B) = \{\phi_z: z \text{ is real; } -1 \leqslant z \leqslant 1\}$. In view of Theorem 2.1 and Lemma 2.3, $p_\infty(f) = \sup_n r_{p_n}(f) = \sup\{|f(z)|: -1 \leqslant z \leqslant 1\} < \infty$ $(f \in B)$ defines greatest continuous C^*-seminorm; and $(\mathscr{E}(B), \tau)$ is the supnorm C^*-algebra $C[-1, 1]$ of all continuous functions on $[-1, 1]$. By [26, Corollary 5.6], for any $f \in B$, the spectrum $\mathrm{sp}(f) = \{f(z): z \in \bar{U}\}$. Thus B is not sb; hence it fails to be a Q-algebra under any topology making it a topological algebra [26, Appendix E].

For $x \in A$, let the hermitian spectral radius of x be defined as $r^h(x) = \sup\{r(\pi(x)): \pi \in R'(A)\}$, $r(\pi(x))$ being the spectral radius of the operator $\pi(x)$ in the C^*-algebra $B(H_\pi)$.

COROLLARY 2.5.

The algebra A has C^*-enveloping algebra iff there exists $p \in K(A)$ and $k > 0$ such that $r^h(x) \leqslant kp(x)$ for all $x \in A$.

Proof. If A has C^*-enveloping algebra, then there exists $p \in K(A)$ and $k > 0$ such that $p_\infty(x) \leqslant kp(x)$ for all x. It follows that $r^h(x) \leqslant kp(x)$ for all x. Conversely, suppose that there exists $p \in K(A)$ and $k > 0$ such that $r^h(x) \leqslant kp(x)$ for all x. Let q be any continuous C^*-seminorm on A. Let $\sigma \in R'_q(A)$. Then, for any $x \in A$, $\|\sigma(x)\|^2 = \|\sigma(x^*x)\| \leqslant r(\sigma(x^*x)) \leqslant r^h(x^*x) \leqslant kp(x^*x) \leqslant kp(x)^2$ giving $\sigma \in R'_p(A)$. Thus $R'_q(A) \subset R'_p(A)$, with the result, for each $x \in A$, $q(x) = r_q(x) \leqslant r_p(x)$. It follows that $r_p(\cdot)$ is the greatest continuous C^*-seminorm on A; and A has C^*-enveloping algebra.

COROLLARY 2.6.

Let A be a hermitian algebra with C^*-enveloping algebra. Then A is a Q-algebra.

Proof. By [26, Theorem 5.2], the hermiticity of A implies that, for each $q \in K(A)$, the Banach* algebra $(A_q, \|\cdot\|_q)$ is hermitian. Hence, by [8, Lemma 41.2], for each $z \in A_q$, the spectral radius in A_q, $r_{A_q}(z) \leqslant r_{A_q}(z^*z)^{1/2} = |z_q|_q$, where $|\cdot|_q$ denotes the Gelfand-Naimark pseudonorm on A_q. Then $m_q(x) = |x_q|_q (x \in A)$ defines a continuous C^*-seminorm on A. By Theorem 2.1, there exists greatest continuous C^*-seminorm $p_\infty(\cdot)$ on A. By [26, Corollary 5.3], for each $x \in A$, the spectral radius in A, $r(x) = \sup\{r_{A_q}(x_q): q \in K(A)\} \leqslant$

$\sup\{m_q(x):q\in K(A)\}\leqslant p_\infty(x)$. By continuity of $p_\infty(\cdot)$, there exists $p\in K(A)$ and $k>0$ such that $r(x)\leqslant p_\infty(x)\leqslant kp(x)$ $(x\in A)$. It follows from [26, Proposition 13.5] (or [3, Theorem 14]) that A is a Q-algebra.

Recall that $P_\alpha(A)$ is the set of all continuous positive linear functionals f on A such that $|f(x)|\leqslant p_\alpha(x)$ for all x. As in [16, Theorem 3.1], the bijective correspondence $f\to f_\alpha:x_\alpha\to f_\alpha(x_\alpha)=f(x)$ $(x\in A)$, identifies $P_\alpha(A)$ with the set $P(A_\alpha)$ of all positive linear functionals (automatically continuous) on the Banach* algebra A_α (having bai $((e_\gamma)_\alpha)$). Let $P_c(A)=\cup_\alpha P_\alpha(A)$. The following identifies $P_c(A)$ intrinsically.

Lemma 2.7. *Let f be a continuous positive linear functional on A. Then $f\in P_c(A)$ iff $|f(x)|^2\leqslant f(x^*x)$ for all $x\in A$.*

Proof. Let there be an $\alpha\in\Delta$ such that $f\in P_\alpha(A)$, so that $|f(x)|\leqslant p_\alpha(x)$ $(x\in A)$. Then, for any $x\in A$, by the continuity of f and Cauchy-Schwarz inequality, $|f(x)|^2=\lim|f(e_\gamma x)|^2\leqslant(\overline{\lim}_\gamma f(e_\gamma e_\gamma^*))f(x^*x)\leqslant(\overline{\lim}\,p_\alpha(e_\gamma e_\gamma^*))f(x^*x)\leqslant(\overline{\lim}\,p_\alpha(e_\gamma)^2)f(x^*x)\leqslant f(x^*x)$. Conversely, assume that $|f(x)|^2\leqslant f(x^*x)$ $(x\in A)$. Since f is continuous and $P=(p_\alpha:\alpha\in\Delta)$ is directed, there exists $k>0$ and $\alpha\in\Delta$ such that $|f(x)|\leqslant kp_\alpha(x)$ $(x\in A)$. Thus for any x, $|f(x)^2|\leqslant f(x^*x)\leqslant kp_\alpha(x^*x)\leqslant kp_\alpha(x)^2$; hence by iterations, $|f(x)|\leqslant k^{1/2^n}p_\alpha(x)$ $(x\in A, n\in\mathbb{N})$. It follows that $f\in P_\alpha(A)\subset P_c(A)$.

For each α, let $B_\alpha(A)$ be the set of all nonzero extreme points of $P_\alpha(A)$. Let $B_c(A)$ be the nonzero extreme points of $P_c(A)$. As in [16], $B_c(A)=\cup_\alpha B_\alpha(A)$. Let $S_c(A)$ denote the set of all continuous C^*-seminorms on A. The following is immediate in view of [16, Lemma 4.1] and Theorem 2.1.

COROLLARY 2.8.

Let A have C^*-enveloping algebra. Then for each $x\in A$, $p_\infty(x)=\sup\{f(x^*x)^{1/2}:f\in P_c(A)\}=\sup\{f(x^*x)^{1/2}:f\in B_c(A)\}=\sup\{p(x):p\in S_c(A)\}$.

COROLLARY 2.9.

The algebra A has C^*-enveloping algebra iff $P_c(A)$ is equicontinuous iff $B_c(A)$ is equicontinuous.

Proof. Let A have C^*-enveloping algebra, so that by Theorem 2.1, the topology τ on $A/s\,\mathrm{rad}\,A$ is determined by the C^*-norm $\|x+s\,\mathrm{rad}\,A\|_{p_\infty}=p_\infty(x)$ $(x\in A)$; and there is $c>0$ and $p\in K(A)$ such that $p_\infty(x)\leqslant cp(x)$ $(x\in A)$. Since the quotient topology t_q on $A/s\,\mathrm{rad}\,A$ induced by the topology t of A is finer than τ, it follows that for a given bai (e_γ) of A, there exists $k>0$ such that $p_\infty(e_\gamma)=\|e_\gamma+s\,\mathrm{rad}\,A\|_{p_\infty}\leqslant k$ for all γ. Let $f\in P_c(A)$. By Corollary 2.8 and the Cauchy-Schwarz inequality, it follows that for any $x\in A$, $|f(x)|=\lim_\gamma|f(e_\gamma x)|\leqslant(\lim_\gamma\sup f(e_\gamma^* e_\gamma)^{1/2})(f(x^*x))^{1/2}\leqslant(\lim_\gamma\sup p_\infty\cdot(e_\gamma^* e_\gamma)^{1/2})p_\infty(x)\leqslant kp_\infty(x)\leqslant kcp(x)$. Thus $P_c(A)$ (hence $B_c(A)$) is equicontinuous. Conversely, let $B_c(A)$ (or $P_c(A)$) be equicontinuous, so that, there exists $p\in K(A)$ and $k>0$ such that $|f(x)|\leqslant kp(x)$ for all $x\in A$, $f\in B_c(A)$. Then the quantity $q(x)=\sup\{f(x^*x)^{1/2}:f\in B_c(A)\}<\infty$ $(x\in A)$ defines greatest continuous C^*-seminorm on A. Thus Theorem 2.1 applies.

COROLLARY 2.10.

Let A be commutative. Then A has C^*-enveloping algebra iff the hermitian Gelfand space

$\mathcal{M}^*(A)$ *is equicontinuous. In this case, for each* $x\in A$, $p_\infty(x)=\sup\{|f(x)|:f\in\mathcal{M}^*(A)\}=$ $^h(x)$.

Remark 2.11. There is an analogy between Q lmc algebras and lmc-*algebras with C^*-enveloping algebras. Corollaries 2.5 and 2.9 correspond to the fact that B is a Q-algebra iff there is a continuous submultiplicative seminorm p on B and $k>0$ such hat $r(x)\leqslant kp(x)$ $(x\in A)$ iff (in commutative case) $\mathcal{M}(A)$ is equicontinuous [3]. Analogous to Theorem 2.1, it holds that a commutative B is a Q-algebra iff B admits greatest continuous submultiplicative seminorm q with square property $q(x^2)=$ $q(x)^2 (x\in A)$. The details will appear elsewhere.

Remark 2.12. It follows from [17, Theorem 4.1] that if A and B have C^*-enveloping algebras, then so does the completed projective tensor product $A\hat{\otimes}_\pi B$; and $\mathscr{E}(A\hat{\otimes}_\pi B)=\mathscr{E}(A)\hat{\otimes}_{\max}\mathscr{E}(B)$, the maximal C^*-tensor product. Also, if I is a closed regular *ideal of A, then A has C^*-enveloping algebra iff I and the completion of A/I have C^*-enveloping algebras.

Example 2.13. The purpose of this example is to exhibit that (a) there exists a commutative Frechet *algebra B and a closed *subalgebra D such that B has C^*-enveloping algebra but D fails to have C^*-enveloping algebra, (b) there exists a commutative non-Q-algebra B with C^*-enveloping algebra such that B is strongly spectrally bounded (ssb) [17], i.e. for some family $Q=(q_\delta)\subset K(B)$ determining the topology of B, $\sup_\delta q_\delta(x)<\infty$ for all $x\in B$. The example is a modification of [8, Example 16, p. 202]. Let C be a complete lmc-*algebra having identity with a $P=(p_\alpha)\subset K(C)$ determining its topology. Let $B=C\oplus C$ with the product topology defined by the seminorms $q_\alpha((x,y))=\max(p_\alpha(x),p_\alpha(y))$. The involution $(x,y)^*=(y^*,x^*)$ makes B a complete unital lmc-*algebra on which $f(z^*z)=0$ $(z\in B)$ for any positive linear functional f on B. Thus $R(B)=\{0\}$, $B=s\,\mathrm{rad}\,B$ and $\mathscr{E}(B)=(0)$, the trivial C^*-algebra.

(i) Let $D=\{(x,x)\in B:x\in C\}$, a closed *subalgebra of B, homeomorphically *isomorphic to A. Take C to be the Frechet *algebra $C(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ with pointwise operations, complex conjugation and with the compact open topology k. The resulting algebra D does not have C^*-enveloping algebra.

(ii) Note that A is ssb (respectively, Q-algebra) iff B is ssb (respectively, Q-algebra). Take C to be the *algebra $C[0,1]$ of all continuous functions of $[0,1]$ with the topology of uniform convergence on all countable compact subsets of $[0,1]$. The resulting algebra B is non-Q, ssb and having C^*-enveloping algebra.

Let f be a positive linear functional, not necessarily continuous, on A. The GNS construction $(\pi_f, D(\pi_f), H_f)$ defines a *homomorphism π_f of A into linear operators (not necessarily bounded) all defined on a dense invariant subspace $D(\pi_f)$ of a Hilbert space H_f as follows: Let $N_f=\{x\in A:f(x^*x)=0\}$, $D(\pi_f)=A/N_f$ with inner product $\langle x+N_f, y+N_f\rangle=f(y^*x)$, $H_f=$ the Hilbert space obtained by completing $D(\pi_f)$, and $\pi_f(x)(y+N_f)=xy+N_f$. Further, each $\pi_f(x)$ is a bounded operator (so that, by extension, $\pi_f(x)\in B(H_f)$), iff f is admissible i.e., for each x, $\sup\{f(y^*x^*xy)/f(y^*y): f(y^*y)\neq 0,\ y\in A\}^{1/2}$ $(=\|\pi_f(x)\|)<\infty$. Also, f is extendible if f can be extended as a positive linear functional on the *algebra A_e obtained by adjoining the identity to A. As a consequence of the presence of a bai on A, every continuous positive functional f on A is extendible and hence admissible by Propositions 3.2 and 3.3 of [2]. Let

$P(A)$ be the set of all positive linear functionals f on A satisfying $|f(x)|^2 \leqslant f(x^*x)\,(x \in A)$. Let $AP(A) = \{f \in P(A) : f \text{ is admissible}\}$, $S(A)$ be the set of all C^*-seminorms on A (not necessarily continuous). For $f \in AP(A)$, $p_f(x) = \|\pi_f(x)\|$ gives a $p_f \in S(A)$. For $x \in A$, define $l(x) = \sup\{p(x) : p \in S(A)\}$, $u(x) = \sup\{p_f(x) : f \in AP(A)\}$, $m(x) = \sup\{f(x^*x)^{1/2} : f \in AP(A)\}$, $s(x) = r(x^*x)^{1/2}$.

Theorem 2.14. (1) *If the algebra A is *sb, then A admits greatest C^*-seminorm $|\cdot|$ (say). In this case,* $|x| = l(x) = u(x) = m(x)$ *for all* $x \in A$.

(2) *Let A be *sb. The following are equivalent.*

(i) *A is hermitian.*

(ii) $s(x) = |x|$ *for all x.*

(iii) $x \to s(x)$ *is a C^*-seminorm on A.*

If, moreover, A is hermitian, then A is sb and $r(x) \leqslant s(x)$ $(x \in A)$.

(3) *Let A be *sb and have C^*-enveloping algebra. The following are equivalent.*

(i) *A is hermitian.*

(ii) $s(x) = p_\infty(x)$ *for all x.*

(iii) $x \to s(x)$ *is a continuous C^*-seminorm on A.*

If A is commutative and hermitian, then $r(x) = p_\infty(x)$ *for all x.*

Lemma 2.15. (1) *The algebra A is hermitian iff A is symmetric.*

(2) *Let* $a = a^*$ *in A with* $r(a) < 1$. *There exists* $x = x^*$ *in A with* $r(x) < 1$ *such that* $2x - x^2 = a$.

(3) *Let* $f \in P(A)$, $b \in A$, $f_b(x) = f(b^*xb)$ $(x \in A)$. *Then the following hold.*

(i) $|f_b(k)| \leqslant r(k) f(b^*b)$ *for all* $k = k^*$ *in A.*

(ii) $|f_b(a)| \leqslant s(a) f(b^*b)$ *for all a in A.*

(4) *Given* $p \in S(A)$, $b \in A$, *there exists* $f \in AP(A)$ *such that* $|f(x)| \leqslant p(x)$ *for all* $x \in A$ *and* $f(b^*b) = p(b^*b)$.

(5) *Let A be Frechet. Each* $p \in S(A)$ *is continuous.*

(6) *Let A be *sb. For all* $p \in S(A)$, $p(x) \leqslant s(x)$ *for all* $x \in A$.

(7) *Let A be *sb and hermitian. The following hold.*

(i) $r(x) \leqslant s(x)$ *for all* $x \in A$.

(ii) $x \to s(x)$ *is a C^*-seminorm on A.*

Proof of the lemma. We prove (2). Let $\mathscr{P}_m(A)$ denote the collection of all families $Q = (q_\delta) \subset K(A)$ such that Q determines the topology of A. Given such a Q, the *subalgebra $B_Q = \{x \in A : \sup_\delta q_\delta(x) < \infty\}$ is a Banach* algebra with the norm $q(x) = \sup_\delta q_\delta(x)$, $x \in B_Q$. Now let $a = a^* \in A$ and $r(a) < 1$. By [19, Theorem 4], there exists a $Q \in \mathscr{P}_m(A)$ such that $a \in B_Q$ and the spectral radius of a in B_Q, $r_{B_Q}(a) < 1$. Ford's square root lemma [8, Proposition 12.11] applied to (B_Q, q) gives $x = x^*$ in B_Q with $2x - x^2 = a$ and $r(x) = r_{B_Q}(x) < 1$. This gives (2). The assertion (5) is a consequence of the automatic continuity of the homomorphism $\pi : A \to A_p$ from a Frechet *algebra with a bai to a C^*-algebra. The remaining assertions can be proved by using inverse limit decomposition of A and using corresponding results for Banach* algebras from [8].

Proof of Theorem 2.14. Given $p \in S(A)$, $b \in A$, there exists $f \in AP(A)$ as in Lemma 2.15(4). Then $p_f(b)^2 = p_f(b^*b) = \|\pi_f(b^*b)\| = \sup\{f(x^*b^*bx) : f(x^*x) = 1\} \geqslant f(b^*b) = p(b)^2$ showing $l(b) \leqslant u(b)$. Thus $u(x) = l(x)$ for all $x \in A$. Similarly,

$p(b)^2 = f(b^*b) \leqslant m(b)^2$, so that $l(x) \leqslant m(x)$ for all x. Let $f \in AP(A)$ and $\xi_f \in H_f$ be a topologically cyclic vector of norm 1 for π_f so that $f(x) = \langle \pi_f(x)\xi_f, \xi_f \rangle$ $(x \in A)$. Then $f(x^*x) = \|\pi_f(x)\xi_f\|^2 \leqslant p_f(x)^2 \leqslant u(x)^2$ implies $m(x) \leqslant u(x)$. Thus, for all x, $l(x) = m(x) = u(x) = |x|$ (say), which is the greatest C^*-seminorm, if it exists. Assuming A to be *sb, one gets $|x| \leqslant s(x)$ for all x, as follows: For $f \in AP(A)$, $x \in A$, the boundedness of $\pi_f(x)$ implies that

$$\begin{aligned} p_f(x) = \|\pi_f(x)\| &= \sup\left\{ \frac{\|\pi_f(x)\eta\|}{\|\eta\|} \,|\, \eta \neq 0 \text{ in } H_f \right\} \\ &= \sup\left\{ \frac{\|\pi_f(x)(y+N_f)\|}{\|y+N_f\|} \,|\, y \in A, f(y^*y) \neq 0 \right\} \\ &= \sup\left\{ \frac{f(y^*x^*xy)^{1/2}}{f(y^*y)^{1/2}} \,|\, y \in A, f(y^*y) \neq 0 \right\} \\ &\leqslant r(x^*x)^{1/2} = s(x) \end{aligned}$$

by Lemma 2.15(3).

(2) follows from Lemma 2.15 and arguments similar to those in Banach* algebras [18, Theorem 41.11].

(3) Let A be *sb and have a C^*-enveloping algebra. Then (ii) implies (i) follows from above (2). Conversely, if A is hermitian, then each A_α is hermitian, hence symmetric by the Shirali-Ford Theorem, and so by the Raikov symmetry criterion [29, Theorem 4.7.21], for all $x \in A$, $r(x^*x) = \sup_\alpha r_{A_\alpha}(x_\alpha^* x_\alpha) = \sup_\alpha[\sup\{f(x_\alpha^* x_\alpha): f \in B(A_\alpha)\}] = \sup_\alpha[\sup \cdot \{f(x^*x): f \in B_\alpha(A)\}] = \sup\{f(x^*x): f \in B_c(A)\} = p_\infty(x)^2$ by Corollary 2.8. This, with Lemma 2.15(6), gives $s(x) = p_\infty(x)$ for all x. That (ii) implies (iii) is immediate and (iii) implies (i) follows from (2). If A is commutative, then using [26, § 5], it is easily seen, as in [8, Theorem 35.3], that A is hermitian iff $\mathscr{M}(A) = \mathscr{M}^*(A)$; and if A is *sb, then this holds iff $r(x^*x) = r(x)^2$ for all x. Thus $x \to r(x) = s(x)$ is a C^*-seminorm dominated by a $p \in K(A)$, as A is also Q-algebra by Corollary 2.6. Thus $r(x) = p_\infty(x)$ for all x.

Remarks 2.16. (1) The pro-C^*-algebra $C[0,1]$ of continuous functions on $[0,1]$ with the topology of uniform convergence on all countable compact subsets of $[0,1]$ admits greatest C^*-seminorm, but fails to admit greatest continuous C^*-seminorm.

(2) It follows from Theorem 2.15 that a *sb Frechet *algebra has C^*-enveloping algebra. This is analogous to the result that a sb Frechet algebra is a Q-algebra [3, Theorem 1].

(3) The Frechet *algebra B of Example 2.4 has C^*-enveloping algebra, it admits greatest C^*-seminorm, but is not *sb (as is exhibited by the function $f(z) = z$ in B).

3. Examples: function algebras

Throughout this section, we consider *algebras of functions with pointwise operations and complex conjugation as the involution (except in Example 3.5).

3.1 Let X be a compact, second countable C^∞-manifold. Let $C^\infty(X)$ be the *algebra of all C^∞-functions on X with the topology of uniform convergence on X of functions and all their derivatives. It is a Frechet sb hermitian Q-algebra, and $\mathscr{E}(C^\infty(X)) = C(X)$, the

supnorm C^*-algebra of all continuous functions on X. If A is a complete lmc-*algebra with C^*-enveloping algebra, then by [17, Corollary 4.3] $\mathscr{E}(C^\infty(X)\hat{\otimes}_\pi A) = \mathscr{E}(C^\infty(X))\hat{\otimes}_\nu \mathscr{E}(A) = C(X)\hat{\otimes}_\varepsilon \mathscr{E}(A) = C(X, \mathscr{E}(A))$, $\mathscr{E}(A)$-valued continuous functions on X.

3.2 Let $(B_k)_{k=1}^\infty$ be a sequence of commutative Frechet lmc-*algebras with identities. Let $A_n = B_1 \oplus \cdots \oplus B_n$ with the product topology and the natural involution. Then $(A_n)_1^\infty$ is an m-compatible sequence, and $A = \cup_{n=1}^\infty A_n$ is an involutive LF-algebra. It is a Q-algebra (hence has C^*-enveloping algebra), if each A_n is a Q-algebra (in particular, each B_k is a Banach* algebra) [26, Proposition 15.8]. In particular, the algebra $C_c(\mathbb{R}^n)$ of continuous function on $\mathbb{R}^n$ with compact supports and with the measure topology [26, Example 3.5], as well as the test function algebras $C_c^k(\mathbb{R}^n)$, $1 \leqslant k \leqslant \infty$ of C^k-functions with compact supports and with the Schwartz topologies are Q-algebras. One has $\mathscr{E}(C_c(\mathbb{R}^n)) = \mathscr{E}(C_c^k(\mathbb{R}^n)) = \mathscr{E}(C_c^\infty(\mathbb{R}^n)) = C_0(\mathbb{R}^n)$, the C^*-algebra of continuous functions on $\mathbb{R}^n$ vanishing at infinity. The Frechet *algebra $s(\mathbb{R}^n)$ of rapidly decreasing C^∞-functions on $\mathbb{R}^n$-with the Schwartz topology in an AE-algebra [12, p. 89], hence is lmc [12, Proposition 1], and $\mathscr{E}(s(\mathbb{R}^n)) = C_0(\mathbb{R}^n)$.

3.3 The constructions, analogous to the one due to Arens [1], also lead to several algebras with C^*-enveloping algebras. For $1 \leqslant p < \infty$, let $AC^p[0,1] = \{f \in C[0,1]$: the derivative f' exists a.e. and $f' \in L^p[0,1]\}$, a Banach* algebra with norm $|f|_p = \|f\|_\infty + (\int_0^1 |f'(t)|^p \mathrm{d}t)^{1/p}$, $\|\cdot\|_\infty$ being the supnorm on $C[0,1]$. Let $AC^\omega[0,1] = \cap_{1\leqslant p<\infty} AC^p[0,1]$, a Frechet *algebra, with topology defined by $f \to |f|_p$, $1 \leqslant p < \infty$; and $AC^\omega[0,1] = \varprojlim_p AC^p[0,1]$. Since $\mathscr{M}(AC^p[0,1]) = [0,1]$ by [29, p. 303]. $\mathscr{M}(AC^\omega[0,1]) = [0,1]$ by [26, Proposition 7.5]. The algebra $AC^\omega[0,1]$ is hermitian Q-algebra, and $\mathscr{E}(AC^\omega[0,1]) = C[0,1]$. One can also consider the Sobolev spaces $W_{p,k}[0,1] = \{f \in C^{k-1}[0,1]: f^{(k-1)} \in AC[0,1]$ and $f^{(k)} \in L^p[0,1]\}$, which are Banach* algebras with norms

$$\|f\|_{p,k} = \sup_{0\leqslant t\leqslant 1} \sum_{l=1}^{k} \frac{|f^{(l)}(t)|}{l!} + \left(\int_0^1 |f^{(k)}(t)|^p \mathrm{d}t \right)^{1/p},$$

and analogously construct the Sobolev-Arens algebras $W_{\omega,k}[0,1] = \cap_{1\leqslant p<\infty} W_{p,k}[0,1]$.

3.4 Let $C_b(\mathbb{R})$ be the C^*-algebra of all bounded continuous functions on $\mathbb{R}$. Let $BV_{\mathrm{loc}}C_b(\mathbb{R}) = \{f \in C_b(\mathbb{R}): f$ is of bounded variation on $[-n,n]$ for all $n = 1,2,3,\ldots\}$, a Frechet lmc *algebra having seminorms $p_n(f) = \|f\|_\infty + V_n(f)$, $V_n(f)$ denoting the total variation of f on $[-n,n]$. One has $\mathscr{E}(BV_{\mathrm{loc}}C_b(\mathbb{R})) = C_b(\mathbb{R})$.

3.5 Let $U = \{z \in \mathbb{C} \mid |z| < 1\}$, $H(U)$ be the algebra, with pointwise operations, of all holomorphic functions on U. Let $A^n(U) = \{f \in H(U): f^{(k)}$ has continuous extension on $\bar{U}$ for all k, $0 \leqslant k \leqslant n\}$, a Banach* algebra with involution $f^*(z) = \overline{f(\bar{z})}$ and norm $\|f\|_n = \sup_{z\in\bar{U}} \Sigma_{k=0}^n (1/k!)|f^{(k)}(z)|$. The Frechet *algebra $A^\omega(U) = \cap_{n=0}^\infty A^n(U)$, with the topology defined by $f \to \|f\|_n$, $n = 0,1,2,\ldots$, is a non-hermitian Q-algebra with $\mathscr{E}(A^\omega(U)) = C[-1,1]$.

4. Segal Algebras

The following is a modification of the definition in [11] tailored for the present set up.

DEFINITION 4.1.

Let $(A, \|\cdot\|)$ be a Banach* algebra with a bai. A *subalgebra B of A is an A-Segal *algebra, if there exists a topology τ on B satisfying the following:

(a) B is a dense *ideal in A.
(b) (B, τ) is a complete lmc *algebra with a bai.
(c) The inclusion $(B, \tau) \to (A, \|\cdot\|)$ is continuous.
(d) The multiplication $(A, \|\cdot\|) \times (B, \tau) \to (B, \tau)$ is continuous.

PROPOSITION 4.2.

*If B is an A-Segal *algebra, then B is a Q-algebra; and $\mathscr{E}(B) = \mathscr{E}(A)$, the enveloping C^*-algebra of A.*

Proof. B being an ideal, $B_{-1} = A_{-1} \cap B$; which is open in (B, τ) by above (c). We show that for continuous (with respective topologies) topologically irreducible *representations, $R'(A) = R'(B)$ via restriction map. By (c), there exists $p_0 \in K(B)(= K(B, \tau))$ such that $\|x\| \leqslant kp_0(x)(x \in B)$, with the result, $\pi \in R'(A)$ implies $\|\pi(x)\| \leqslant \|x\| \leqslant kp_0(x)\,(x \in B)$ and $\pi|_B \in \mathrm{R}'(B)$ in view of (a). Let $\pi \in R'(B)$. Let $\xi \in H_\pi$ be any topologically cyclic vector for π so that $H_\pi =$ closed span of $\{\pi(B)\xi\}$. Let (e_γ) be a bai for B. Let $x \in A$, $y \in B$. Then $xy \in B$; and $e_\gamma y \to y$ in (B, τ). By (d), $xe_\gamma y \to xy$ in (B, τ); hence $\|\pi(xe_\gamma y) - \pi(xy)\| \to 0$. Since (e_γ) is τ-bounded, $\|\pi(xe_\gamma)\| \leqslant M_x < \infty$. Thus $\|\pi(xy)\xi\| = \lim_\gamma \|\pi(xe_\gamma)\pi(y)\xi\| \leqslant M_x \|\pi(y)\xi\|$. Thus the bounded linear operator $\tilde{\pi}(x):\pi(y)\xi \to \pi(xy)\xi$ defines $\tilde{\pi} \in R'(A)$, $\tilde{\pi}|_B = \pi$. Thus $R'(A) = R'(B)$. Let $P = (p_\alpha) \subset K(B, \tau)$ determine τ on B. Then, for any $y \in B$, $p_\infty(y) = \sup_\alpha r_\alpha(y) = \sup_\alpha\{\sup_{\pi \in R'\alpha(B)} \|\pi(y)\|\} = \sup\{\|\pi(y)\| : \pi \in R'(B)\} = \sup\{\|\pi(y)\| : \pi \in \mathrm{R}'(A)\} \leqslant \|y\| \leqslant kp_0(y)$; and the greatest continuous C^*-seminorm $p_\infty(\cdot)$ on (B, τ) is the restriction of the Gelfand-Naimark pseudonorm (denoted by $p_\infty(\cdot)$ only). Finally, we show that $\mathscr{E}(B) = [B/(N(p_\infty) \cap B), \|\cdot\|_{p_\alpha}]^{\sim} = (A/(N(p_\infty)), \|\cdot\|_{p_\alpha})^{\sim} = \mathscr{E}(A)$. The map $\phi(x + (N(p_\infty) \cap B)) = x + N(p_\infty)$ is a well defined *isomorphism of $B/(N(p_\infty) \cap B)$ into $A/N(p_\infty)$. Thus $\mathscr{E}(B)$ is a C^*-subalgebra of $\mathscr{E}(A)$. Let $z \in \mathscr{E}(A)$. There exist sequences (x_n) in A, (y_n) in B such that

$$\|x_n + N(p_\infty) - z\|_{p_x} < \frac{1}{2^{n+1}}, \quad \|x_n - y_n\| < \frac{1}{2^{n+1}}.$$

Then,

$$\begin{aligned}\|y_n + N(p_\infty) - z\|_{p_x} &\leqslant \|x_n + N(p_\infty) - z\|_{p_\infty} + \|y_n - x_n\|_{p_\infty} \\ &\leqslant \frac{1}{2^{n+1}} + \|y_n - x_n\| \\ &< \frac{1}{2^n} \to 0 \text{ as } n \to \infty\end{aligned}$$

showing that $\mathscr{E}(B)$ is dense in $\mathscr{E}(A)$.

Examples: Convolution algebras. For various group algebras on locally compact groups, we take convolution multiplication and the involution $f^*(g) = \Delta(g^{-1})\overline{f(g^{-1})}$, Δ being the modular function. Throughout, $\|\cdot\|_p$ denotes the usual norm on L^p-space.

4.3 For a locally compact abelian group G, take $A = L^1(G)$, $B = L^\omega(G) := \{f \in L^1(G): f \in L^p(G) \text{ for all } p, 1 < p < \infty\}$, the topology τ on B is determined by submultiplicative seminorms $p_k(f) = \|f\|_1 + \|f\|_k$, $k = 2, 3, 4, \ldots$. Then $\mathscr{E}(B) = C^*(G)$, the group C^*-algebra of G.

4.4 For $A = L^1(\mathbb{R})$, $B = \{f \in L^1(\mathbb{R}): f \in C^\infty(\mathbb{R})$ and the derivative $f^{(n)} \in L^1(\mathbb{R})$ for all $n = 1, 2, \ldots\}$. The topology τ on B is defined by $p_k(f) = \|f\|_1 + \|f^k\|_1$, $k = 0, 1, 2, \ldots$. That $p_k(f * g) \leqslant p_k(f) p_k(g)$ is a consequence of the identity $(f * g)^{(k)} = f^{(k)} * g = f * g^{(k)}$. Then B is an A-Segal *algebra.

4.5 Let G be a compact group. Let $(S, |\cdot|)$ be a Banach* algebra with a bai. For $1 \leqslant p < \infty$, $B^p(G, S)$ be the Banach* algebra of functions $f: G \to S$ with $|f|_p = [\int |f(g)|^p \mathrm{d}\mu]^{1/p} < \infty$.

Then $B^p(G, S)$ can be realized as a suitable completed tensor product $L^p(G) \hat{\otimes}_p S$, with the norm $\eta_p(\cdot)$ defined by taking a finite tensor $f = \Sigma x_i \otimes y_i$, as $\eta_p(f) = [\int |\Sigma x_i(g) y_i|^p \mathrm{d}\mu]^{1/p}$. By [23, Proposition 7.10], for all p, $\mathscr{E}(B^p(G, S)) = C^*(G) \hat{\otimes}_{\min} \mathscr{E}(S)$. Taking $A = B^1(G, S)$, $B = B^\omega(G, S) = \cap_{1 \leqslant p < \infty} B^p(G, S) = \lim_p B^p(G, S)$ with the topology of $\|\cdot\|_p$-convergence for each p, $\mathscr{E}(B^\omega(G, S)) = C^*(G) \hat{\otimes}_{\min} \mathscr{E}(S)$.

4.6 Let $(A, \|\cdot\|)$ be a commutative hermitian Banach* algebra with a bai. Let μ be a positive regular Borel measure on $\mathscr{M}(A) = \mathscr{M}^*(A)$. For $1 \leqslant p < \infty$, let $A_p(\mu) = \{x \in A: \hat{x} \in L^p(\mathscr{M}(A), \mu)\}$, $\hat{x}$ denoting the Gelfand transform of x. It is a Banach* algebra with norm $\|x\|_{A_p} = \|x\| + \|\hat{x}\|_p$. For the A-Segal *algebra $B = \cap_{1 \leqslant p < \infty} A_p(\mu)$, $\mathscr{E}(B) = \mathscr{E}(A)$. In particular, for a locally compact abelian group G with dual group $\hat{G}$ and Haar measure μ on $\hat{G}$, consider $A = L^1(G)$, $B = \{f \in L^2(G)$: Fourier transform $\hat{f}$ is in $L^p(\hat{G}, \mu)$, $1 \leqslant p < \infty\}$.

5. Topological algebras with bases and Köthe sequence algebras

Let ω denote the *algebra of all scalar sequences $(a_n)_1^\infty$ with pointwise operations and complex conjugation. Let $Q \subset \omega$ be a Köthe power set, i.e. Q satisfies (i) for each $a \in Q$, $a_n \geqslant 0$ for all n; (ii) for each $a \in Q$, $b \in Q$, there exists $c \in Q$ such that $a_n \leqslant c_n$, $b_n \leqslant c_n$ for all n; and (iii) for each n, there exists $a \in Q$ satisfying $a_n > 0$. We assume $a_1 \neq 0$ for all $a \in Q$. Further, let Q satisfy G_∞-property, i.e. (iv) for each $a \in Q$, $a_n \leqslant a_{n+1}$ for all n; (v) for each $a \in Q$, there exists $d \in Q$ such that $a_n^2 \leqslant d_n$ for all n. Köthe space of infinite type ([22], [32, p. 203]) is the complete locally convex space $\Lambda_\infty(Q) = \{x = (x_n) \in \omega: p_\alpha(x) = \Sigma |x_n| a_n < \infty \text{ for all } a \in Q\}$ with the locally convex Köthe topology t defined by the family $\Gamma = (p_\alpha: a \in Q)$ of seminorms. It so turns out [4] that $(\Lambda_\infty(Q), t)$ is a complete lmc-*algebra which is a *subalgebra of $(\ell^1, \|\cdot\|_1)$ and which is a Q-algebra. (Note that a complete commutative continuous inverse Q-algebra is lmc [31].) Further, it is hermitian. For the present purpose, we note the following.

PROPOSITION 5.1.

(i) $\Lambda_\infty(Q)$ *is an* ℓ^1*-Segal *algebra.*
(ii) $\mathscr{E}(\Lambda_\infty(Q)) = c_0$, *the* C^**-algebra of all null sequences.*

(i) can be easily checked; whereas (ii) follows from a more general result to follow. The following important particular case of $\Lambda_\infty(Q)$ we shall need.

5.2 ($\Lambda_\infty[\theta_n]$): For an increasing sequence (θ_n) of positive numbers, $Q = \{(k^{\theta_n})_{n=1}^\infty : k = 1, 2, \ldots\}$ gives the algebra $\Lambda_\infty(Q)$ denoted by $\Lambda_\infty[\theta_n]$ called power series space of infinite type [32, p. 204]. The algebra $s = \{x : \omega : \Sigma_{n=1}^\infty n^k |x_n| < \infty$ for all $k = 1, 2, \ldots\}$ of rapidly decreasing sequences is $\Lambda_\infty[\theta_n]$ taking $\theta_n = \log n$, $n = 1, 2, \ldots$ [32, p. 205].

Recall [5, §1] that an orthogonal basis on a topological algebra A is a basis (f_n) for A such that $f_n f_m = \delta_{nm} f_n$ for all n, m in $\mathbb{N}$, δ_{nm} being the Kronecker delta. The algebra $\Lambda_\infty(Q)$ admits $f_n = (\delta_{\mathrm{nm}})_{m=1}^\infty$ as an orthogonal basis.

PROPOSITION 5.3.

*Let A be a complete hermitian lmc-*algebra with C^*-enveloping algebra. Let A admit an orthogonal basis consisting of hermitian elements. Then A is a Q-algebra; and $\mathscr{E}(A)$ is *isomorphic to the C^*-algebra c_0.*

Proof. Let (f_n) be an orthogonal basis for A, $f_n^* = f_n$ for all n. Then (f_n) is a Schauder basis and A is commutative [5], [20]. Let ϕ_n be the coefficient functional defined by f_n viz expanding $x \in A$ as $x = \Sigma_n x_n f_n$, $x_n \in \mathbb{C}$, $\phi_n(x) = x_n$. Then the Gelfand space $\mathscr{M}(A) = \{\phi_n\} = \mathscr{M}^*(A)$ by hermiticity; and $\phi : A \to \omega$, $\phi(x) = (x_n)$ is a *isomorphism of A onto a *subalgebra K of ω; which is continuous for the topology of pointwise convergence on K [20, Corollary 1.3]. We identify A with K algebraically. By [26, Corollary 5.6], for all $x \in A$, the spectrum $sp(x) = \{\phi_n(x)\} = \{x_n\}$; and by Corollary 2.10 and Thorem 2.1, $p_\infty(x) = \sup_n |x_n| = \|x\|_\infty = r(x) < \infty$ for all x. Thus $A \subset \ell^\infty$. By Corollary 2.6, A is a Q-algebra. Further, A contains the set of all finitely many nonzero sequences. Also, A cannot have identity, otherwise A has to be ω with pointwise convergence [20, Theorem 2.1], which is not an algebra with C^*-enveloping algebra. It follows that $\mathscr{E}(A)$, which is the completion of $(A, \|\cdot\|_\infty)$, contains c_0. On the other hand, $p_\infty(\cdot)$ being a continuous C^*-seminorm on A (in the topology of A), $x^{(n)} = \Sigma_1^n x_k f_k \to x = \Sigma_1^\infty x_k f_k$ in A implies that $|x_{n+1}| \leqslant \sup_{k>n} |x_k| = p_\infty(x - x^{(n)}) \to 0$ as $n \to \infty$. Thus $x = (x_n) \in c_0$, $\mathscr{E}(A) \subset c_0$ with the result, $\mathscr{E}(A) = c_0$.

Example 5.4. Let $A = C^\infty(\Gamma)$, the convolution algebra of all C^∞-functions on the circle Γ with involution $u^*(z) = u(z^{-1})$. By [14, p. 48], for any $u \in C^\infty(\Gamma)$, the Fourier series expansion $u = \Sigma_{-\infty}^\infty \hat{u}(n) e^{int}$ gives a sequence $(\hat{u}(n)) \in s(\mathbb{Z})$, = two sided rapidly decreasing sequences. The map $\phi : C^\infty(\Gamma) \to s(\mathbb{Z})$, $\phi(u) = (\hat{u}(n))$ establishes a *isomorphism of $C^\infty(\Gamma)$ onto s, which is a homeomorphism for the (usual) Frechet C^∞-topology on $C^\infty(\Gamma)$ and Frechet Köthe topology on $s(\mathbb{Z})$ [30, Theorem 5.1]. Now, $s(\mathbb{Z})$ is a Q-algebra and $\mathscr{E}(s(\mathbb{Z})) = c_0$. Thus, via ϕ, $C^\infty(\Gamma)$ is a complete Q lmc-*algebra with $\mathscr{E}(C^\infty(\Gamma)) = \{\mu \in \mathrm{PM}(\Gamma) : (\hat{\mu}(n)) \in c_0\}$, where $\mathrm{PM}(\Gamma)$ is the convolution algebra of all pseudo measures on Γ, isomorphic to ℓ^∞ via Fourier expansion [14, §12.11].

Example 5.5. For the open unit disc U, let $H^p(U)$ be the Hardy space, for $1 < p < \infty$. The Banach space $(H^p(U), \|\cdot\|_p)$ is a Banach* algebra with Hadamard product

$$(f * g)(x) = (1/2\pi i) \int_{|z|=r} f(z) g(xz^{-1}) z^{-1} \mathrm{d}z, \; |x| < r < 1,$$

having involution $f^*(z) = \overline{f(\bar{z})}$. The sequence $e_n(z) = z^n$ is an orthogonal basis for $H^p(U)$ [20, Example 3]. Thus, the Hardy-Arens algebra $H^\omega(U) = \cap_{1<p<\infty} H^p(U)$ is a

Frechet lmc-*algebra with basis (e_n), the topology being the topology of $\|\cdot\|_p$-convergence for each p, $1<p<\infty$. The coefficient functionals ϕ_n are $\phi_n(f)=f^{(n)}(0)/n!$, the nth Taylor coefficient of f(exactly as in [15, Example-3.2(ii)] for the *algebra $H(U)$); and for any $f\in H^\omega(U)$, $\mathrm{sp}(f)=\{\phi_n(f)\}$. It is easily seen that $H^\omega(U)$ is hermitian, and $p_\infty(f)=\sup_n(|f^{(n)}(0)|/n!)\leqslant\|f\|_p$ $(p>1)$ is the greatest continuous C^*-seminorm.

Example 5.6. Let E be the Frechet space of all entire functions of one complex variable with compact open topology. It is a topological *algebra with Hadamard product and the involution $f^*(z)=\overline{f(\bar{z})}$, admitting orthogonal basis $e_n=z^n, n\in\mathbb{N}$. The mapping $\phi:E\to\omega$, $\phi(\Sigma x_n e_n)=(x_n)$ is a *isomorphism of E onto the sequence algebra $\Lambda_\infty[n]$. Also, ϕ is a homeomorphism for the respective topologies on E and $\Lambda_\infty[n]$ [32, p. 206]. Thus $\mathscr{E}(E)$ is *isomorphic to c_0.

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$C(K, X)$ as an M-ideal in $WC(K, X)$

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Abstract. In this paper we study the classes of Banach spaces X for which the space of continuous X-valued functions forms an M-ideal in the space of weakly continuous functions. We also study a lifting problem for weakly continuous functions.

Keywords. M-ideals; weakly continuous functions; Schur property.

1. Introduction

For an infinite dimensional Banach space X and an infinite compact Hausdroff space K let $C(K, X)$ denote the Banach space of X-valued continuous functions on K equipped with the supremum norm and let $WC(K, X)$ denote the space of functions that are continuous when X has the weak topology, equipped with the supremum norm. In a recent work Diestel *et al* [4] show that any element of $WC(K, X)$ is Bochner μ-integrable w.r.t. each regular Borel probability measure μ on K and thus identify $C(K, X)^*$ as a subspace of $WC(K, X)^*$ and obtain the decomposition

$$WC(K, X)^* = C(K, X)^* \oplus C(K, X)^\perp$$

via the restriction map
The question raised by them is "when is the sum above an l_1 sum?"

Let us recall that a subspace $J \subset X$ is said to be an M-deal if $J^\perp \oplus_1 N = X^*(l_1 - \text{sum})$ for some closed subspace N. Also if $J \subset X$ is such that $J^\perp$ is the kernel of a norm one projection P in X^* and if J is an M-ideal then $N = \text{Range}$ P. Hence, in terms of M-structure theory (see [1] for all relevant definitions) an equivalent formulation is that "When is $C(K, X)$ an M-ideal in $WC(K, X)$?"

In this paper we look at this question and obtain some positive and negative results. Our first theorem disposes off the trivial situation and more.

Theorem 1. *The following statements are equivalent.*

1. *X has the Schur property*
2. *$C(K, X) = WC(K, X)$ for any K*
3. *For any K, every element of $WC(K, X)$ attains its norm on K*
4. *For some K, every element of $WC(K, X)$ attains its norm*

Proof. 1⇒2: Let $f \in WC(K, X)$. Since X has the Schur property, $f(K)$ is a norm compact subset of X and hence on $f(K)$ weak and norm topologies coincide. Therefore f is norm continuous.

$2 \Rightarrow 3 \Rightarrow 4$ are clear.
$4 \Rightarrow 1$: Suppose X fails the Schur property. Assume w.l.o.g $\exists$ *a* $y_n \in X$, $\|y_n\| = 1$ and $y_n \to 0$ weakly.
Fix any $\alpha \in l^\infty$ with

$$\sup_n |\alpha(n)| = 1 > |\alpha(n)| \forall n$$

Let $x_n = \alpha(n) y_n$ then $x_n \to 0$ weakly. Fix a distinct sequence $k_n \in K$ and a pairwise disjoint sequence of open sets U_n with $k_n \in U_n$. Choose $f_n \in C(K)$, $0 \leqslant f_n \leqslant 1$ and $f_n(k_n) = 1$, $f_n = 0$ on $K \backslash U_n$.

Define $g: K \to X$ by $g(k) = \Sigma f_n(k) x_n$. Clearly g is well defined and $\|g\| = 1$. To see that g is weakly continuous, note that for any $x^* \in X^*$, $x^* \circ g = \Sigma x^*(x_n) f_n$ and since $x^*(x_n) \to 0$ the RHS is a continuous function. To obtain the required contradiction we now show that g fails to attain its norm on K. Suppose for some k_0, $\|g(k_0)\| = 1$. Let n_0 be such that $k_0 \in U_{n_0}$, then $g(k_0) = f_{n_0}(k_0) x_{n_0} = f_{n_0}(k_0) \alpha_{n0} y_{n0}$. Since $\|y_{n0}\| = 1$ and $f_{n0}(k_0) \leqslant 1, |\alpha_{n0}| < 1$, we get a contradiction. Therefore X has the Schur property.

In spite of the decomposition of $WC(K, X)^*$ the precise nature of its elements is far from being clear, hence the following corollary is of some interest. Let $\partial e X_1^*$ denote the extreme points of the dual unit ball and for any $k \in K$, let $\delta(k)$ denote the Dirac measure at k. It is well known that

$$\partial e C(K, X)_1^* = \{\delta(k) \oplus x^* : k \in K,\ x^* \in \partial e X_1^*\}.$$

Note that for any function $f: K \to X$,

$$(\delta(k) \oplus x^*)(f) = x^*(f(k)).$$

COROLLARY.

X has the Schur property iff

$$\partial e WC(K, X)_1^* = \{\delta(k) \oplus x^* : k \in K,\ x^* \in \partial e X_1^*\}.$$

Proof. Suppose X fails the Schur property and

$$\partial e WC(K, X)_1^* = \{\delta(k) \oplus x^* : k \in K,\ x^* \in \partial e X_1^*\}.$$

Let g be the function constructed during the proof of $4 \Rightarrow 1$ above. By the Hahn-Banach theorem,

$$1 = \|g\| = \Lambda(g) \text{ for some } \Lambda \in \partial e WC(K, X)_1^*.$$

By our assumption, $\Lambda = \delta(k) \oplus x^*$ for some $k \in K$ and $x^* \in \partial e X_1^*$.
Now $1 = \Lambda(g) = x^*(g(k)) \leqslant \|g(k)\| \leqslant 1$. Hence $\|g(k)\| = 1$ contradicting the fact that g fails to attain its norm.

From now on we assume that $C(K, X)$ is a proper subspace of $WC(K, X)$. For Banach spaces X, Y let us denote by $\mathscr{K}(X, Y) =$ space of compact operators, $\mathscr{F}(X, Y) =$ space of weakly compact operators and $\mathscr{L}(X, Y) =$ space of bounded operators.

For any index set Γ, let $\oplus_0^\Gamma X$ denote X-valued functions defined on Γ and

vanishing at ∞ and let $\oplus_{\infty}^{\Gamma} X$ denote the space of X-valued bounded functions defined on Γ. Both these spaces are equipped with the supremum norm.

It is well-known that

$$\oplus_0^{\Gamma} X \text{ is an M-ideal in } \oplus_{\infty}^{\Gamma} X$$

for any Banach space X and index set Γ. Our first result is based on the following easy observation about M-ideals.

Observation. For Banach spaces X, Y, Z with $Z \subset Y \subset X$, if Z is an M-ideal in X then Z is an M-ideal in Y.

PROPOSITION 1.

For any discrete set Γ, *and for any compact* K,

$$C(K, c_0(\Gamma)) \textit{ is an M-ideal in } WC(K, c_0(\Gamma)).$$

Proof. Let us note the canonical identification

$$C(K, c_0(\Gamma)) = \oplus_0^{\Gamma} C(K)$$

and

$$WC(K, c_0(\Gamma)) \subset \oplus_{\infty}^{\Gamma} C(K)$$

via evaluation at elements of Γ. Since $\oplus_0^{\Gamma} C(K)$ is an M-ideal in $\oplus_{\infty}^{\Gamma} C(K)$, using the observations mentioned above we get that $C(K, c_0(\Gamma))$ is an M-ideal in $WC(K, c_0(\Gamma))$.

In [10] the authors study a class of Banach spaces Y (the so called M_{∞}-spaces) with the property, $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$ for any Banach spaces X, our next result involves subspaces of such a space Y.

Theorem 2. *Let* K *be any compact extremally disconnected space and let* X *be a closed subspace of an* M_{∞}*-space then* $C(K,X)$ *is an* M-*ideal in* $WC(K,X)$.

Proof. Case i: Suppose K is the Stone-Čech compactification $\beta(\Gamma)$ of some discrete space Γ. It is easy to identify $C(\beta(\Gamma), X)$ as $\mathscr{K}(l^1(\Gamma), X)$ by restricting the functions to Γ and the same mapping allows one to identify $WC(\beta(\Gamma), X)$ as a closed subspace of $\mathscr{L}(l^1(\Gamma), X)$. In view of our observation, the result is proved once we note that $\mathscr{K}(l^1(\Gamma), X)$ is an M-ideal in $\mathscr{L}(l^1(\Gamma), X)$. That this is indeed the case can be proved by using arguments identical to the ones given in the proof of Proposition 2.9 in [8].

Case ii: Let K be any compact extremally disconnected space. By well known results in topology (see [6]), there exist a discrete set Γ, an into homeomorphism $\psi: K \to \beta(\Gamma)$ and a continuous onto map $\phi: \beta(\Gamma) \to K$ such that $\phi \circ \psi =$ identity on K. Let Φ denote the canonical isometry $f \to f \circ \phi$ taking function spaces on K isometrically into function spaces on $\beta(\Gamma)$ and let P denote the norm one projection $g \to \Phi(g \circ \phi)$ on the appropriate spaces.

By Case (i) $C(\beta(\Gamma), X)$ is an M-ideal in $WC(\beta(\Gamma), X)$. We shall verify the restricted 3 ball property for $C(K,X) \subset WC(K,X)$ to conclude that it is an M-ideal (see [1]). Let $f_i \in C(K,X)_1$, $1 \leqslant i \leqslant 3$, $g \in WC(K,X)_1$ and $\varepsilon > 0$. Since $C(\beta(\Gamma), X)$ is an M-ideal in $WC(\beta(\Gamma), X)$, applying the restricted 3 ball property to $\Phi(f_i)$, $\Phi(g)$ we get a

$h' \in C(\beta(\Gamma), X)$ such that $\| \Phi(g) + \Phi(f_i) - h' \| \leqslant 1 + \varepsilon \forall i$. Since P is a projection of norm one

$$\| \Phi(g) + \Phi(f_i) - P(h') \| \leqslant 1 + \varepsilon$$

$$\text{i.e. } \| \Phi(g) + \Phi(f_i) - \Phi(h' \circ \psi) \| \leqslant 1 + \varepsilon \text{ or}$$

$$\| g + f_i - h' \circ \psi \| \leqslant 1 + \varepsilon \forall i$$

Now $h' \circ \psi \in C(K, X)$. Hence $C(K, X)$ is an M-ideal in $WC(K, X)$.

Remark. Whether the above theorem is valid for any compact space K is not known. The properties of M_∞-spaces and their subspaces seem to indicate that this should be so.

Related to the above ideas is a question of lifting weakly compact sets. We are interested in the following two situations.

(a) X is a Banach space, $Y \subset X$ is a closed subspace and $\pi: X \to X/Y$ is the quotient map. Given a weakly compact set K in X/Y and $\varepsilon > 0$ there is a weakly compact set $\tilde{K}$ in X such that $\pi(\tilde{K}) = K$ and $\mathrm{Sup}_{\tilde{K}} \| \, \| \leqslant (1 + \varepsilon) \sup_K \| \, \|$.

(b) For a compact K and $f \in WC(K, X/Y)$, $\varepsilon > 0$ there is a $g \in WC(K, X)$ such that $\pi \circ g = f$ and $\| g \| \leqslant (1 + \varepsilon) \| f \|$.

Let us note that this is trivial when X/Y has the Schur property and a quotient map from a $l^1(\Gamma)$ onto a Banach space X does the lifting in (a) only when X has the Schur property (ie in general there is no weakly continuous cross-section map for π).

Examples

1) The authours of [13] show that if Y is a reflexive subspace of a Banach space X, π has lifting as in (a).
2) Let T denote the unit circle and H_0^1 the Hardy space in $L^1(T)$, then a classical theorem in analysis (see [11]) says that π has lifting as in (a). Note that the norm-restrictions are valid since $\tilde{K}$ is the weak closure of image of K under the nearest point cross-section map in this case.
3) X a Banach space, $Y \subset X$ be an L^1-predual. Consider $\pi: X^* \to X^*/Y^\perp$.
Let K be any compact set and let $f \in WC(K, Y^*)$. Define $T: Y \to C(K)$ by $T(y)(k) = f(k)(y)$. It is well known that T is a weakly compact operator and $\| T \| = \| f \|$. Since $Y \subset X$ and Y is an L^1-predual by Theorem 6.1 of [9], $\exists$ a weakly compact operator $\tilde{T}: X \to C(K)$, extending T and such that $\| \tilde{T} \| = \| T \| = \| f \|$. Now $g = (\tilde{T})^* \circ \delta$ (where $\delta: K \to C(K)^*$ the Dirac map) is the necessary weakly continuous lifting.

PROPOSITION 2.

Let X be a Banach space and let $Y \subset X$ be a closed subspace. $B \Rightarrow A$, and $A \Rightarrow B$ for compact extremally disconnected spaces. When B holds and if $C(K,X)$ is an M-ideal in $WC(K, X)$ then the same is true of $C(K, X/Y)$ in $WC(K, X/Y)$.

Proof. $B \Rightarrow A$ is clear.

Let K be compact, extremally disconnected. As in the proof of case [ii] of Theorem 2, get a discrete set Γ and mappings $\psi: K \to \beta(\Gamma)$, $\phi: \beta(\Gamma) \to K$ with $\phi \circ \psi = \text{identity}$.

Given $f\in WC(K,X/Y)$, $\varepsilon>0$ since $f\circ\phi\in WC(\beta(\Gamma), X/Y)$ by property (A) $(f\circ\phi)(\beta(\Gamma))$ can be lifted and hence we can define a $g'\in WC(\beta(\Gamma),X)\ni\|g'\|\leqslant(1+\varepsilon)\|f\|$ and $\pi\circ g'=f\circ\phi$.
Put $g=g'\circ\psi$, $g\in WC(K,X)$,

$$\|g\|\leqslant(1+\varepsilon)\|f\|$$

and for $k\in K$,

$$\pi(g(k))=\pi(g'(\psi(k)))$$
$$=f(\phi(\psi(k)))=f(k)$$

so that $\pi\circ g=f$.
Proof of the rest of the proposition can be completed as in Theorem 2 using the "restricted 3-ball" characterization of M-ideals.

Problem. Does $A\Rightarrow B$?

Even though we do not have a complete description of situations when $C(K,X)$ is an M-ideal in $WC(K,X)$, our last proposition shows that they exhibit properties similar to c_0-spaces.

PROPOSITION 3.

If $C(K,X)$ is an M*-ideal in $WC(K,X)$ then $0\in\overline{\partial eX_1^*}$ (Closure taken in the w*-topology).*

Proof. Let us observe that

$$WC(K,X)_1^*=\overline{CO}(\delta(k)\oplus x^*:k\in K,\ x^*\in\partial eX_1^*)$$

(w*-closed convex hull), since the functionals on the RHS determine the norm. If $WC(K,X)^*=C(K,X)^*\oplus_1 C(K,X)^\perp$ then choose

$$\Lambda\in C(K,X)^\perp\cap\partial eWC(K,X)_1^*.$$

By Milman's converse to the Krein-Milman theorem ([3])$\Lambda\in\{\delta(k)\oplus x^*\colon k\in K, x^*\in\partial eX_1^*\}^{-w^*}$.
Let

$$\Lambda=\lim_\alpha\delta(k_\alpha)\oplus x_\alpha^*,\ k_\alpha\in K,\ x_\alpha^*\in\partial eX_1^*.$$

For any $x\in X$ considered as constant function in $C(K,X)$

$$0=\Lambda(x)=\lim_\alpha x_\alpha^*(x)$$

Therefore $0\in\overline{\partial eX_1^*}$

Negative results.
As before these observations are based on the corresponding facts known for operator spaces. An observation due to Saatkamp [12] in operator theory says that when $\mathscr{K}(X,Y)\neq\mathscr{L}(X,Y)$, $\mathscr{K}(X,Y)$ is not an M-summand in $\mathscr{L}(X,Y)$. Similar argument

works to show that $C(K, X)$ is not an M-summand in $WC(K, X)$. Hence, a standard procedure now to show that $C(K, X)$ is not an M-ideal is to notice when $C(K, X)$ has the intersection property (I.P, See [2], [5]) and then appeal to Theorem 4.3 of [2] to conclude that $C(K, X)$ is not an M-ideal in $WC(K, X)$.

It has been observed in [5] that when X has the I.P, $C(K, X)$ has the I.P and examples of Banach spaces X with I.P include $C(K)$ spaces, reflexive Banach spaces and more generally spaces with the Radon-Nikodým property, spaces with a non-trivial l^p-summand for $p < \infty$ (see [2]). In all these situations $C(K, X)$ is not an M-ideal in $WC(K, X)$.

For a dual space X^*, identifying

$$C(K, X^*) = \mathscr{K}(X, C(K)),\ WC(K, X^*) = \mathscr{F}(X, C(K))$$

when X^* fails the I.P, since X^* has a copy of c_0 (see [2])., arguments given during the Proof of Proposition 2.2 [8] work to show that if $\mathscr{K}(X, C(K))$ is an M-ideal in $\mathscr{F}(X, C(K))$ then $\mathscr{K}(l^1, C(K))$ is an M-ideal in $\mathscr{F}(l^1, C(K))$. But as we have noted before $\mathscr{K}(l^1, C(K)) = C(\beta(N), C(K))$ and $\mathscr{F}(l^1, C(K)) = WC(\beta(N), C(K))$ and since $C(K)$ has the I.P this cannot happen. So for no dual space X^*, $C(K, X^*)$ can be an M-ideal in $WC(K, X^*)$.

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Note

After submitting this paper for publication, I have received an expanded version of [4] from Professor J. Diestel. The authors of [14] now have also some answers to the M-ideal question. Their approach is different from the M-structure theoretic approach that I have taken. The purpose of this note is to illustrate this point. The following characterization of the class M_∞ appears in [16]. $X \in M_\infty$ iff there is a net K_α in the unit ball of $\mathscr{K}(X)$ such that

$$K_\alpha x \to x \forall x \in X \text{ and } K_\alpha^* x^* \to x^* \forall x^* \in X^*$$

and for any $\varepsilon > 0$ there is an $\alpha_0 \ni \forall \alpha > \alpha_0$.

$$\| K_\alpha x + (I - K_\alpha) y \| \leqslant (1+\varepsilon) \max\{ \|x\|, \|y\| \} \text{ for all } x, y \in X (\dagger).$$

Note that if a K_α satisfying (†) is a projection then one has

$$\|x\| \leqslant (1+\varepsilon)\max\{ \|K_\alpha x\|, \|x - K_\alpha x\| \}$$

and

$$\|K_\alpha\| \leqslant 1 + \varepsilon$$

and

$$\|I - K_\alpha\| \leqslant 1 + \varepsilon.$$

Projections satisfying these conditions are called almost L^∞-projections (see [15]).

Now let us recall from [14] the definition of Schur approximation property. A Banach space X has the Schur approximation property (SAP for shor) if for any compact set $K \subset X$ and $\varepsilon > 0$ there is a projection P with range (P) having the Schur property such that

$$\|x - Px\| < \varepsilon \forall x \in K$$

$$\|P\| \leqslant 1 + \varepsilon, \|I - P\| \leqslant 1 + \varepsilon$$

and

$$\|x\| \leqslant (1+\varepsilon)\max\{ \|Px\|, \|x - Px\| \}.$$

PROPOSITION 4.

Let K be a compact Hausdorff space and let $X \in M_\infty$. $C(K,X)$ is an M-*ideal in $WC(K,X)$.*

Proof. As before we shall verify the restricted 3-ball property.

Let $f \in WC(K,X)$, $f_i \in C(K,X)$ be in their respective unit balls and let $\varepsilon > 0$. Put

$K^{\sim} = \cup_{i=1}^{3} f_i(K)$ and use (†) to get a compact operator K_α such that

$$\| K_\alpha x + (I - K_\alpha) y \| \leqslant (1 + \varepsilon) \max\{ \|x| \, \|y\| \} \forall x, \quad y \in X$$

and

$$\| K_\alpha x - x \| < \varepsilon \forall x \in K^{\sim}.$$

Put $g = K_\alpha \circ f$. Clearly $g \in C(K, X)$.
For any $k \in K$

$$\|f_i(k) + f(k) - g(k)\| \leqslant \|(I - K_\alpha)(f(k)) + K_\alpha(f_i(k)\| + \|f_i(k) - K_\alpha(f_i(k))\|$$
$$\leqslant (1 + \varepsilon) \max\{ \|f(k)\|, \|f_i(k)\| \} + \varepsilon \leqslant 1 + 2\varepsilon \forall i$$

Remark 1. It follows from the results in Chapter VI of [16] that for any $Y \in M_\infty$ of infinite dimension, every infinite dimensional subspace has an isomorphic copy of c_0. Consequently only finite dimensional subspaces here have the Schur property. So for a $X \subset Y$, $Y \in M_\infty$ the SAP for X already implies the bounded approximation property. However in Theorem 2 above we have made no assumptions about approximation property and such spaces X without the bounded approximation property are known to exist.

Remark 2. An argument similar to the one above gives an M-structure theoretic proof of "$C(K, X)$ is an M-ideal in $WC(K, X)$ when X has the SAP," which is Theorem 7 of [14].

PROPOSITION 5.

If a Banach space X has the Schur property then every M-*ideal in X is an M-summand and there are only finitely many M-summands.*

Proof. Key fact is that X and none of its subspaces have an isomorphic copy of c_0. So if $M \subset X$ is an M-ideal and infinite dimensional then since M has no copy of c_0, M must be an M-summand see [2]. Of course when M is finite dimensional it is already an M-summand, see [1].

An example of a space with the SAP mention in [4] is a c_0 direct sum of spaces with the Schur property. We now show

PROPOSITION 6.

If X has the SAP with M-projections then X is isometric to a c_0 direct sum of spaces with the Schur property.

Proof. Here we consider the maximal function module representation of X ([1]). Then the base spaces have the Schur property and in view of Proposition 5 M-projections correspond to multiplication operator by indicator functions of finite sets, one concludes that X is isometric to a c_0 direct sum of spaces with the Schur property.

Remark. The above formulation and proof are inspired by Proposition 6.5 and its proof in [17].

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On the zeros of a class of generalised Dirichlet series—XI

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Abstract. A sufficiently large class of generalised Dirichlet series is shown to have lots of zeros in $\sigma > 1/2$. Some examples are (i) $\zeta'(s) - a$ (a any complex constant) (ii) $\alpha - \zeta(s) - \Sigma_{n=0}^{\infty}((n+\sqrt{2})^{-s} - (n+1)^{-s})$ (where α is any positive constant) and (iii) $\alpha + \Sigma_{n=1}^{\infty}(-1)^n(\log n)^{\lambda} n^{-s}$ (where λ is any real constant $> 1/2$ and α any complex constant). Here as is usual we have written $s = \sigma + it$.

Keywords. Zeros; generalised Dirichlet series; Riemann zeta-function.

1. Introduction

In paper [1] of this series we considered zeros of $G(s) = \Sigma_{n=1}^{\infty} a_n n^{-s}$ (under fairly general conditions. We have changed the notation for $F(s)$ to $G(s)$ to avoid a clash of notation later) in the rectangle

$$\{\sigma \geqslant \tfrac{1}{2} - \delta, \quad T \leqslant t \leqslant 2T\}, \tag{1}$$

where $\delta = \delta(T) \to 0$ as $T \to \infty$, and as usual $s = \sigma + it$. The only restrictive condition was something like $\Sigma |a_p|^2 \gg x/\log x$, (the sum being over all primes p subject to $x < p \leqslant 2x$) for all large x and what was irksome was the condition $a_1 \neq 0$. The main object of the present paper is to relax the condition $a_1 \neq 0$ to $a_1 = 0, \ldots, a_{n_0} = 0$ and $a_{n_0+1} \neq 0$ where $n_0 (\geqslant 0)$ is an integer constant. Of course we can (as we do) assume $n_0 \geqslant 1$ since the case $n_0 = 0$ is considered in the paper $X^{[1]}$ of this series. Also the condition involving a_p was designed to include $\zeta(s)$; but if we strengthen the lower bound to say $\Sigma |a_p|^2 \gg x(\log x)^2$ then we can prove that $G(s)$ has at least one zero in

$$\{\sigma > \tfrac{1}{2}, \quad T \leqslant t \leqslant 2T \tag{2}$$

provided only that $|G(s)|$ does not exceed a fixed power of T (assuming T to be sufficiently large). Also by using ideas of this paper and those of [7] it is possible to prove that Riemann hypothesis implies that if $q = [\alpha(\log\log T)^{1/2}]$ (where $\alpha > 0$ is a constant) then

$$\lim_{T\to\infty} \inf \left\{ \frac{1}{T} \int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2/q} \mathrm{d}t \right\} \geqslant \exp(\alpha^{-2}). \tag{3}$$

(We may also formulate a result for $1/H\int_T^{T+H}(\cdots)dt$ where $T \geqslant H \gg \log\log T$). The first of these results follows from a routine application of the method of $X^{[1]}$ (except when $a_1 = 0$ in which case the method of the present paper succeeds) while the second follows from the following observation. Consider $G(s)$ where the a_n are multiplicative over square-free integers n. Then the coefficient of $(p_1 \cdots p_k)^{-s}(p_1, \cdots, p_k$ distinct primes) in $(G(s))^{1/q}$ is the same as in

$$\left(1+\frac{a_{p1}}{p_1^s}\right)^{1/q}\left(1+\frac{a_{p2}}{p_2^s}\right)^{1/q}\cdots\left(1+\frac{a_{pk}}{p_k^s}\right)^{1/q}$$

i.e. $q^{-k}a_{p1}a_{p2}\cdots a_{pk}$. We have then to use the Hardy-Ramanujan theorem as in [7]. We do not give further details of the proof of these results. Instead we define a property P_q of a Dirichlet series $G(s) = \Sigma_{n=1}^{\infty} b_n \mu_n^{-s}$ where $\{b_n\}$ is any sequence of complex numbers and $\{\mu_n\}$ is any sequence of real numbers with $b_1 = \mu_1 = 1$, $\mu_1 < \mu_2 < \mu_3 < \cdots$ and $1/C \leqslant \mu_{n+1} - \mu_n \leqslant C$ where $C(\geqslant 1)$ is an integer constant. We assume that the series for $G(s)$ converges absolutely for some complex number s.

DEFINITION

Let $q(\geqslant 2)$ be an integer. We say that $G(s)$ has the property P_q if there exists a constant $\delta > 0$ and a positive integer $n^* = n^*(\delta)$ (n^* not divisible by q) both depending on $G(s)$ such that $G(s)$ can be continued analytically in

$$\{\sigma \geqslant \tfrac{1}{2} + \delta, \quad T \leqslant t \leqslant 2T\} \tag{4}$$

and has $\gg T$ zeros all of order n^* in this rectangle.

Remarks. Also we consider functions like $\log\zeta(s) - \alpha$ where α is any complex constant. These have singularities but continuable in $\sigma \geqslant 1/2$. We prove that $\log\zeta(s) - \alpha$ has the property P_2 (if we allow analytic continuation except on horizontal lines which contain singularities). *In what follows n^* may depend on T; but n^* will be bounded above by a constant depending only on δ.*

Accordingly our theorems which illustrate our method are

Theorem 1. *The function $\zeta'(s) - \alpha$ has the property P_2 for every complex constant α.*

Theorem 2. *The function log $\zeta(s) - \alpha$ has the property P_2 (in the sense explained in the remark above) for every complex constant α.*

Theorem 3. *The function $G(s) = \alpha - \Sigma_{n=0}^{\infty}(n + \sqrt{2})^{-s}$ has the property P_2 for every positive real constant α.*

Theorem 4. *Let $\lambda(> 1/2)$ be any constant. Then $G(s) = \alpha + \Sigma_{n=1}^{\infty}(-1)^n(\log n)^{\lambda} n^{-s}$ has the property P_2 for every complex constant α.*

Theorem 5. *The function $G(s) = \alpha + \Sigma_{n=1000}^{\infty}(-1)^n(\log\log n)^{3/4} n^{-s}$ has the property P_q (for some integer $q = q(\delta)$) for every complex constant α.*

Remarks. More general results will be found in the later sections of this paper. It is possible to deal with the zeros in $\{\sigma \geqslant 1-\delta, T \leqslant t \leqslant 2T\}$ in a somewhat general setting. These questions will be taken up elsewhere. We would like to remark that our results hold good for zeros of Dirichlet polynomials like $\Sigma_{n \leqslant T} a_n \mu_n^{-s}$ and $\Sigma_{n \leqslant T^{1000}} a_n \mu_n^{-s}$ (with conditions on $\{a_n\}$ of a fairly general nature and somewhat restrictive conditions on $\{\mu_n\}$).

The previous history of Theorems 1 and 2 is well-known and due to many authors. (For references see [8]. Of great relevance here is the work of Bohr and Jessen [4, 5]. But both our methods and results seem to be new).

2. A conjecture and its proof in special cases

We believe that the following conjecture is true (at least in a modified form). In [2] we have proved it in some special cases and these will be used in the present paper. (We stipulate that certain constants shall be integers only for a technical reason which is not serious). We quote from the paper just cited.

Conjecture. Let $1=\mu_1<\mu_2<\cdots$ *be any sequence of real numbers with* $1/C \leqslant \mu_{n+1}-\mu_n \leqslant C$ *where* $C(\geqslant 1)$ *is an integer constant and* $n=1,2,3,\cdots$. *Let us form the sequence* $1=\lambda_1<\lambda_2<\cdots$ *of all possible (distinct) finite power products of* $1=\mu_1,\mu_2,\cdots$ *with non-negative integral exponents. Let* $s=\sigma+it$, $H(\geqslant 10)$ *a real parameter, and* $\{a_n\}(n=1,2,3,\cdots)$ *with* $a_1=1$ *be any sequence of complex numbers (possibly depending on* H*) such that* $F(s)=\Sigma_{n=1}^{\infty} a_n \lambda_n^{-s}$ *is absolutely convergent at* $s=B$ *where* $B(\geqslant 3)$ *is an integer constant. Suppose that* $F(s)$ *can be continued analytically in* $(\sigma \geqslant 0, 0 \leqslant t \leqslant H)$ *and that there exist* T_1, T_2 *with* $0 \leqslant T_1 \leqslant H^{3/4}$, $H-H^{3/4} \leqslant T_2 \leqslant H$ *such that for some* $K(\geqslant 30)$, *there holds*

$$\max_{\sigma \geqslant 0}(|F(\sigma+iT_1)|+|F(\sigma+iT_2)|) \leqslant K. \tag{5}$$

Finally let $\Sigma_{n=1}^{\infty}|a_n|\lambda_n^{-B} \leqslant H^A$ *where* $A(\geqslant 1)$ *is an integer constant. Then there exists a* $\delta_1(>0)$ *(depending only on* A,B,C*) such that for all* $H \geqslant H_0(A,B,C)$ *there holds*

$$\frac{1}{H}\int_0^H |F(it)|^2 \mathrm{d}t \geqslant \frac{1}{2}\sum_{\lambda_n \leqslant H^{\delta_1}} |a_n|^2, \tag{6}$$

provided that $H^{-1}\operatorname{loglog} K$ *does not exceed a small positive constant.*

Remark. We have used the symbol δ_1 (in place of δ) so that it should not clash with the δ already introduced. Also we recall that $1/2$ can be replaced by a quantity ~ 1 (as $H \to \infty$) and whenever we have succeeded in proving this conjecture we have proved it in this stronger form.

We now quote the corollaries to the main theorem of [2].

COROLLARY 1.

Let $\mu_n = n$. *Then the conjecture is true.*

COROLLARY 2.

Let $n_0(\geqslant 2)$ be an integer constant, and $\mu_n = (n_0 + n - 1)/(n_0)$. Then the conjecture is true.

COROLLARY 3.

Let $\beta > 0$ be an algebraic constant, and $\mu_n = ((n+\beta)/(1+\beta))$. Then the conjecture is true. (The conjecture is also true for the choice $\mu_1 = 1$, $\mu_n = n + \beta - 1$ for $n > 1$).

Remark. It is possible to state a slightly more general corollary than Corollary 3. But we do not state it since our ambition is to prove a sufficiently general result.

3. Two important observations

We record the observations as two lemmas.

Lemma 1. *Let $\mu_n = (n_0 + n - 1)/(n_0)$ and $G(s) = \Sigma_{n=1}^{\infty} b_n \mu_n^{-s}$ be absolutely convergent for some complex s. Then, we have, for any integer $q > 0$ and σ large enough,*

$$(G(s))^{1/q} = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \tag{7}$$

where the λ_n are formed as in the conjecture, $a_1 = 1$, and further whenever $n_0 + n - 1$ is prime $|a_n| = q^{-1}|b_n|$, and so the RHS of (6) is

$$\geqslant \frac{1}{2q^2} \sum_{\mu_n \leqslant H^{\delta_1}} |b_n|^2 \tag{8}$$

where the sum is restricted to those n for which $n = 1$, and also to those n for which $n_0 + n - 1$ is prime.

Proof. It is sufficient to check that if p is a prime $\geqslant n_0 + 1$, the equality

$$\frac{\ell_1 \cdots \ell_k}{n_0^k} = \frac{p}{n_0}$$

where $\ell_1, \cdots, \ell_k$ are integers $\geqslant n_0 + 1$, is not possible except when $k = 1$ and $\ell_1 = p$. This is trivial since p has to divide at least one ℓ_j say ℓ_1. Now

$$n_0^{k-1} = \left(\frac{\ell_1}{p}\right) \ell_2 \cdots \ell_k \geqslant (n_0 + 1)^{k-1}$$

which is impossible unless $k = 1$.

Lemma 2. *Let $G(s) = 1 - \Sigma_{n=2}^{\infty} b_n \mu_n^{-s}$* where *$b_n$ are real and non-negative and the series involved converges for some complex s. Then for any integer $q(> 0)$ and σ large enough, we have,*

$$(G(s))^{1/q} = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$$

where the λ_n are as in the conjecture, $a_1 = 1$ and further for $n \geqslant 2$, $a_n \leqslant 0$ and $-a_n \geqslant b_n q^{-1}$ wherever $\lambda_n = \mu_n$.

4. Proof of theorems 1, 3, 4 and 5

We sketch the proof in a general setting. Note that after an easy normalisation the functions in question look like $G(s) = \Sigma_{n=1}^{\infty} b_n \mu_n^{-s}$, where $b_1 = 1$, $\{\mu_n\}$ as in any of the Corollaries 1, 2 or 3 (of § 2), which converges absolutely for some complex number s and is analytically continuable in $\sigma > 1/2$. It is easy to see that, for $\sigma = 1/2 + \delta$,

$$\frac{1}{T}\int_T^{2T} |G(\sigma + it)|^2 \mathrm{d}t \ll \sum_{n=1}^{\infty} |b_n|^2 n^{-1-2\delta} = V(2\delta), \quad \text{say.} \tag{9}$$

From this and the fact that the absolute value of an analytic function at the centre of a circle is majorised by its mean-value over the disc enclosed by it, it follows that

$$\sum_{|I| = H} \max_{s \in ([(1/2)+\delta, \infty) \times I)} |G(s)|^2 \ll \delta^{-2} V(\delta)\, T \tag{10}$$

where I runs over all disjoint intervals of length H into which $[T, 2T]$ can be divided with a suitable meaning at the end points. We assume that $H \leqslant T^{1/2}$ and that H is a large enough function of δ. From (10) it follows that

$$\#\{I : |I| = H, \max |G(s)|^2 \geqslant \delta^{-3} V(\delta) H\} \ll \delta T/H. \tag{11}$$

Let $q \geqslant 2$ be an integer. In order to obtain the lower bound

$$\frac{1}{H}\int_I |G(s)|^{2/q} \mathrm{d}t \gg \sum_{\lambda_n \leqslant H^{\delta_1}} |a_n|^2 n^{-1-2\delta}, \quad (s = \tfrac{1}{2} + \delta + it), \tag{12}$$

we have to check the condition that $H^{-1} \log\log K$ shall not exceed a small positive constant. In (12) $\{a_n\}$ are defined by

$$F(s) = (G(s))^{1/q} = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}.$$

If we assume that in $[\frac{1}{2} + \delta, \infty] \times I$, $F(s)$ is regular (i.e. $G(s)$ has no zeros of order not divisible by q) then (12) holds if H exceeds a large constant depending on δ since we can take $K = \delta^{-3} V(\delta) H$ provided we omit the intervals counted in (11). Also

$$\#\left\{I : \frac{1}{H}\int_I |G(s)|^2 \mathrm{d}t \geqslant \eta^{-1} V(2\delta)\right\} \ll \frac{\eta T}{H}, \quad (s = \tfrac{1}{2} + \delta + it), \tag{13}$$

where $\eta > 0$ is a small constant.

Hence we have $\gg TH^{-1}$ intervals I (with $|I| = H$) for which (12) holds and also

$$\frac{1}{H}\int_I |G(s)|^2 \mathrm{d}t \leqslant \eta^{-1} V(2\delta). \tag{14}$$

We now show that each of the rectangles $[\frac{1}{2} + \delta, \infty] \times I$ (for these I) must contain

a zero of $G(s)$ of order not divisible by q (if we impose a suitable condition on $V(2\delta)$ and $V(\delta)$). Otherwise from (12) and (14) we must have

$$\left(\frac{D_1}{q^2}\sum_{n\leqslant H^{\delta_1}}' |b_n|^2 n^{-1-2\delta}\right)^q < D_2\eta^{-1}V(2\delta) \tag{15}$$

where $D_1 > 0$, $D_2 > 0$, and η are independent of T, $\bar{H}$, q and δ. Also the accent restricts the sum as in (8). If the $\{\mu_n\}$ are as in Corollary 3 we end up with

$$\left(\frac{D_1}{q^2}\sum_{n\leqslant H^{\delta_1}} |b_n|^2 n^{-1-2\delta}\right)^q \leqslant D_2\eta^{-1}V(2\delta). \tag{16}$$

Since we are interested in finding some $H = H(\delta)$ contradicting (15) and (16) we can as well contradict

$$\left(\frac{D_1}{q^2}\sum_{n=1}^{\infty}{}' |b_n|^2 n^{-1-2\delta}\right)^q \leqslant D_2\eta^{-1}V(2\delta) \tag{17}$$

for proving Theorems 1,4 and 5. To prove Theorem 3 we have to contradict

$$\left(\frac{D_1}{q^2}\sum_{n=1}^{\infty} |b_n|^2 n^{-1-2\delta}\right)^q \leqslant D_2\eta^{-1}V(2\delta). \tag{18}$$

It is a trivial matter to check that (17) and (18) are false for the particular cases in question. This completes the proofs of Theorems 1,3,4 and 5 except for the remark concerning n^* (for this see § 7).

5. Some generalisations

It is plain that we can prove analogues of Theorem 1 (also Theorem 2 as will be seen) to $\zeta''(s)$, $\zeta'''(s)$, $\cdots$, derivatives of L-functions and also to derivatives of the zeta and L-functions of any quadratic field. We can also prove the analogues of Theorems 3,4 and 5 to more general Dirichlet series. We are particularly interested in (stating the analogue for) a class of functions in which we were interested in [3]. We proceed to recall their definition.

Let $\chi(n)(n = 1, 2, 3, \cdots)$ be a periodic sequence of complex numbers not all zero (if the period is k we require that there is at least one integer n with $(n, k) = 1$ and $\chi(n) \neq 0$) such that the sum $\Sigma\chi(n)$ extended over a period is zero. Let $f(x)$ be a positive real valued function of x defined for $x \geqslant 1$ such that for every fixed $\varepsilon > 0$, $f(x)x^{\varepsilon}$ is increasing and $f(x)x^{-\varepsilon}$ is decreasing for all $x \geqslant x_0(\varepsilon)$. Let $\{d_n\}$ $(n = 1, 2, 3, \cdots)$ be a sequence of complex numbers satisfying $f(n) \ll |d_n| \ll f(n)$ and for all $X \geqslant 1$ we should have

$$\sum_{X\leqslant n\leqslant 2X} |d_{n+1} - d_n| \ll f(X).$$

The functions that we wish to consider are

$$G(s) = \sum_{n=1}^{\infty} \chi(n) d_n n^{-s}.$$

Let us suppose that the expression

$$E(\delta)=\left(\sum_{n=1}^{\infty}(f(n))^2 n^{-1-2\delta}\right)^{1/2}\left(\sum_{n=1}^{\infty}f(n)(\log(n+1))^{-1}n^{-1-2\delta}\right)^{-1} \tag{19}$$

tends to zero as $\delta\to 0$. Then, we have

Theorem 6. *The function $G(s)-\alpha$ has the property P_2 for every complex constant α.*

Proof. This follows from the arguments of § 5 and § 7. We have only to observe that $f(x)\ll f(2x)\ll f(x)$ and that $\pi(x)\asymp x/\log x$.

Remark. We can also state a similar theorem for the property $P_q(q=q(\delta))$.

6. Proof of theorem 2

The proof is not very much different from the one sketched in § 4. Note that we have the density theorem that $N(\sigma, T)$ defined by

$$\#\{\rho:\zeta(\rho)=0 \ \mathrm{Re}\,\rho\geqslant\sigma, |\mathrm{Im}\,\rho|\leqslant T\}$$

is $O(T^{\nu(1-\sigma)}(\log T)^5)$ where $\nu=3/(2-\sigma)$ due to Ingham [6] (see also page 236 of [8]). The O-constant is independent of σ and T. In view of this theorem the number of t-intervals I of constant length $H=H(\delta)$, satisfying $T\leqslant t\leqslant 2T$ such that $[\frac{1}{2}+\delta/2,\infty)\times I$ is zero free is $\sim$ T/H. This and the remark in § 7 are enough for the proof of Theorem 2.

Remark 1. We may also remark that the analogue of Theorem 2 is true for the logarithm of a finite power product (with complex exponents not all zero) of ordinary L-functions or L-functions of a fixed quadratic field since for these L-functions the function $N(\sigma, T)$ is $O(T^{\nu'(1-\sigma)}(\log T)^{C_0})$ where $\nu'=4/(3-2\sigma)$ and C_0 is an absolute constant. The O-constant depends on the modulii of the characters.

Remark 2. Starting from Theorem 2 one may deduce easily the following.

Theorem 7. *The function $\zeta(s)-e^{\alpha}$ has the property P_2 for every complex constant α.*

7. Completion of proofs

We have proved that for the functions in question the number of distinct zeros in $\{\sigma\geqslant\frac{1}{2}+\delta,\ T\leqslant t\leqslant 2T\}$ whose orders are not divisible by q is $\gg$ T. But by a slight variant of the considerations of the proof we can secure that the $\gg TH^{-1}$ intervals I selected for the contradiction have the property that in the rectangles $[\frac{1}{2}+\delta/2,\infty]\times I$ the functions are bounded by a function of δ. By Jensen's theorem it follows that the number of zeros (in these rectangles) counted with multiplicity is bounded. Thus the orders of the $\gg T$ zeros as proved already in § 4, § 5 and § 6 are bounded by a function of δ alone. Hence (by classifying these zeros according to their

orders) we see that $\gg T$ zeros (in at least one class) have the same order (a fixed integer not divisible by q). This completes the proof of all our assertions.

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POST-SCRIPT. The condition $E(\delta)\to 0$ as $\delta\to 0$ see ((9)) can be proved under various choices of $f(n)$. For example let $(\log n)^2 \leqslant f(n) \leqslant \exp((\log n)^{0\cdot 1})$. Then $E(\delta)\to 0$ as $\delta\to 0$. To see this we begin with a

Lemma. Let $f(n)(n=1,2,3,\cdots)$ *be any sequence of positive real numbers such that* $(\log f(n))(\log n)^{-1}\to 0$ *as* $n\to\infty$. *For any* $\delta>0$ *put*

$$Q_1=\sum_{n=1}^{\infty}(f(n))^2 n^{-1-2\delta},\quad Q_2=\sum_{n=1}^{\infty}(f(n))^2(\log(n+1))^{-1}n^{-1-2\delta},$$

and

$$Q_3=\sum_{1\leqslant \exp(Q_1^{1/4})}(f(n))^2 n^{-1-2\delta}.$$

If $Q_1-Q_3\leqslant \frac{1}{2}Q_1$ *and* $Q_1\geqslant(1/\varepsilon)^2$, $(0<\varepsilon<\frac{1}{2})$, *then* $Q_1\ll \varepsilon Q_2^2$.

Proof. We have

$$\begin{aligned}Q_2 &\geqslant \sum_{1\leqslant \exp(Q_1^{1/4})}(f(n))^2(\log(n+1))^{-1}n^{-1-2\delta}\\ &\gg Q_1^{-1/4}Q_1(\text{with an implied absolute constant, since } Q_3\geqslant \tfrac{1}{2}Q_1)\end{aligned}$$

i.e. $Q_2^2\gg Q_1^{3/2}\gg(1/\varepsilon)Q_1$ since $Q_1\geqslant(1/\varepsilon)^2$.

This completes the proof of the lemma.

COROLLARY.

Let $(\log n)^2\leqslant f(n)\leqslant \exp((\log n)^{0\cdot 1})$. *Then* $E(\delta)\to 0$ *as* $\delta\to 0$.

Proof. In this case $Q_1 \geqslant \Sigma_{n=1}^{\infty}(\log n)^4 n^{-1-2\delta} \asymp \delta^{-5} \geqslant (1/\varepsilon)^2$ if δ is sufficiently small.

We have only to prove that $Q_1 - Q_3 \leqslant \frac{1}{2}Q_1$. Let d be any positive constant. We will show that $Q_4 = \Sigma_{n \geqslant \exp(d\delta^{-1\cdot 25})}(f(n))^2 n^{-1-2\delta}$ tends to zero as $\delta \to 0$. For $n \geqslant \exp(d\delta^{-1\cdot 25})$, we have

$$\begin{aligned}\frac{n^{2\delta}}{(f(n))^2} \geqslant \frac{n^{2\delta}}{\exp(2(\log n)^{0\cdot 1})} &\geqslant \exp\{2\delta \log n - 2(\log n)^{0\cdot 1}\}\\ &\geqslant \exp\{(\log n)^{0\cdot 1}(2\delta(\log n)^{0\cdot 9} - 2)\}\\ &\geqslant \exp\{(\log n)^{0\cdot 1}(2\delta(d^{0\cdot 9})(1/\delta)^{1\cdot 125} - 2\}\\ &\geqslant (\log n)^2 \text{(for all } n \text{ exceeding an absolute constant if } \delta \text{ is small enough).}\end{aligned}$$

Thus $Q_4 \to 0$ as $\delta \to 0$. This proves the corollary completely since $\Sigma_{n=2}^{\infty} n(\log n)^{-2}$ is convergent. (For the validity of $E(\delta) \to 0$ clearly we can impose $(\log n)^{R_1} \leqslant f(n) \leqslant \exp((\log n)^{R_2})$ where $R_1(> 3/2)$ and $R_2(< 1 - 4(2R_1 + 1)^{-1})$ are constants).

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On L^1-convergence of a modified cosine sum

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Abstract. In this paper a generalization of a theorem of Kumari and Ram [4] has been proved by considering the class $S(\delta)$ instead of the class S.

Keywords. L^1-convergence; convex sequence; modified cosine sum.

1. Introduction

A sequence (a_k) is said to be convex if $\Delta^2 a_k \geqslant 0$, where $\Delta^2 a_k = \Delta(\Delta a_k)$ and $\Delta a_k = a_k - a_{k+1}$, and quasi-convex if

$$\sum_{k=1}^{\infty} (k+1)|\Delta^2 a_k| < \infty. \tag{1}$$

A sequence (a_k) of positive numbers is said to be quasi-monotone if $k\Delta a_k \geqslant -\alpha a_k$ for some positive number α and δ-quasi-monotone if $a_k \to 0$, $a_k > 0$ ultimately and $\Delta a_k \geqslant -\delta_k$, where (δ_k) is a sequence of positive numbers [2]. The concept of quasi-convex sequence was generalized by Sidon [5] in the following manner:

A sequence (a_k) is said to belong to class S, or $a_k \in S$, if $a_k \to 0$ as $k \to \infty$, and there exists a sequence of numbers (A_k) such that

$$\begin{aligned}
&\text{(a)} \qquad A_k \downarrow 0, \quad k \to \infty,\\
&\text{(b)} \qquad \sum_{k=1}^{\infty} A_k < \infty, \qquad\qquad (2)\\
&\text{(c)} \qquad |\Delta a_k| \leqslant A_k, \quad \text{for all } k.
\end{aligned}$$

This class S of sequences has been further generalized to the class S^* and $S(\delta)$ by Singh and Sharma [6] and Zaini and Hasan [7], respectively:

$a_k \in S^*$ if (2) holds with the condition (a) replaced by,

(a') (A_k) is quasi-monotone.

$a_k \in S(\delta)$ if (2) holds with the condition (a) replaced by,

(a'') (A_k) is δ-quasi-monotone and $\Sigma k\delta_k < \infty$.

Thus, in view of the above definitions it is obvious that $S \subset S^* \subset S(\delta)$. Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{3}$$

be a cosine series and satisfy $a_k = o(1)$, $k \to \infty$. Let the partial sum of (3) be denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$.

Recently Kumari and Ram [4] introduced the modified cosine sum as

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta(a_j/j) k \cos kx, \tag{4}$$

and obtained its L^1-convergence by proving the following theorem.

Theorem A. *Let (a_k) in (3) belong to the class S. If $\lim_{n\to\infty}|a_{n+1}| \log n = 0$, then $\|f - g_n\| = o(1)$, $n \to \infty$.*

The aim of this paper is to generalize Theorem A by considering the class $S(\delta)$ instead of the class S. Now, we shall prove the following theorem.

Theorem. *Let (a_k) occurring in (3) belong to class $S(\delta)$. If $\lim_{n\to\infty}|a_{n+1}| \log n = 0$, then $\|f - g_n\| = o(1), n \to \infty$.*

3. We need the following lemmas for the proof of our theorem.

Lemma 1. ([3]). *If $|c_k| \leqslant 1$, then*

$$\int_0^{\pi} \left| \sum_{k=0}^{n} c_k \frac{\sin(k+1/2)x}{2\sin\frac{x}{2}} \right| dx \leqslant C(n+1),$$

where C is a positive absolute constant.

Lemma 2. ([7]). *Let (a_k) be a δ-quasi-monotone sequence with $\Sigma k\delta_k < \infty$. If $\Sigma a_k < \infty$, then $\Sigma(k+1)\Delta a_k < \infty$.*

Proof of the theorem

We have

$$\begin{aligned} g_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \{\Delta(a_k/k) + \Delta(a_{k+1}/(k+1)) + \cdots + \Delta(a_n/n)\} \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left\{\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right\} \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx \\ &= S_n(x) - \frac{a_{n+1}}{n+1} \tilde{D}'_n(x), \quad \text{say.} \end{aligned}$$

Now, making use of Abel's transformation and Lemma 1, we get

$$
\begin{aligned}
\int_0^\pi |f(x)-g_n(x)|\,\mathrm{d}x &\leqslant \int_0^\pi \left|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\right| \mathrm{d}x + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| \mathrm{d}x \\
&= \int_0^\pi \left|\sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x)\right| \mathrm{d}x + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| \mathrm{d}x \\
&= \int_0^\pi \left|\sum_{k=n+1}^{\infty} \Delta A_k \sum_{j=0}^{k} \Delta a_j \frac{1}{A_j} D_j(x)\right| \mathrm{d}x \\
&\quad + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| \mathrm{d}x \\
&\leqslant \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \left|\sum_{j=0}^{k} \Delta a_j \frac{1}{A_j} D_j(x)\right| \mathrm{d}x \\
&\quad + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| \mathrm{d}x \\
&\leqslant C \sum_{k=n+1}^{\infty} (k+1)\Delta A_k + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| \mathrm{d}x. \qquad (5)
\end{aligned}
$$

Since $\Sigma(k+1)\Delta A_k < \infty$, by Lemma 2, the first term in (5) tends to zero as $n \to \infty$. On the other hand, by Zygmund's Theorem ([1], p. 458)

$$
\begin{aligned}
\int_0^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| \mathrm{d}x &\leqslant \int_{-\pi}^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| \mathrm{d}x = \frac{|a_{n+1}|}{n+1}\int_{-\pi}^\pi |\tilde{D}'_n(x)|\,\mathrm{d}x \\
&= C|a_{n+1}| \int_{-\pi}^\pi |\tilde{D}'_n(x)|\,\mathrm{d}x \sim |a_{n+1}|\log n \qquad (6)
\end{aligned}
$$

since $\int_{-\pi}^\pi |\tilde{D}'_n(x)|\,\mathrm{d}x$ behaves like $\log n$.

The conclusion of the theorem now follows from (5) and (6).

COROLLARY.
If (a_n) occurring in (3) belongs to the class $S(\delta)$ and $\lim_{n\to\infty}|a_{n+1}|\log n = 0$, then $\|f - S_n\| = o(1)$, $n \to \infty$.

Proof. We notice that

$$
\begin{aligned}
\int_{-\pi}^\pi |f(x)-S_n(x)|\,\mathrm{d}x &= \int_{-\pi}^\pi |f(x)-g_n(x)+g_n(x)-S_n(x)|\,\mathrm{d}x \\
&\leqslant \int_{-\pi}^\pi |f(x)-g_n(x)|\,\mathrm{d}x + \int_{-\pi}^\pi |g_n(x)-S_n(x)|\,\mathrm{d}x \\
&\leqslant \int_{-\pi}^\pi |f(x)-g_n(x)|\,\mathrm{d}x + \int_{-\pi}^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| \mathrm{d}x.
\end{aligned}
$$

Since $\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - g_n(x)|\,\mathrm{d}x = 0$, by our theorem and $\int_{-\pi}^{\pi} \left|\frac{a_{n+1}}{n+1} \tilde{D}'_n(x)\right| \mathrm{d}x$ behaves like $|a_{n+1}|\log n$ by Zygmund's Theorem cited above, for large values of n, the conclusion of the corollary follows.

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Addendum to the paper "generalised parabolic sheaves on an integral projective curve"

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In [1] we constructed moduli spaces of generalised parabolic sheaves (GPS in short) of rank n on an integral projective curve, the generalised parabolic structure being at finitely many (smooth) points of the curve. The method of construction was a natural generalisation of the method of Simpson [2] which uses the well-known embedding of Quot scheme into a Grassmannian. The method of [2] was generalised by M S Narasimhan and T R Ramadas to construct moduli of (generalised) parabolic sheaves of rank 2 (on an integral projective curve) with generalised parabolic structure at a point y and usual parabolic structure at finitely many points y_j different from y [Appendix, [3]). By oversight the reference to this interesting paper was not given in [1].

We would also like to point out the following correction to §2, [1]. In (2–1, [1]) one should have "a point $q \in \tilde{Q}$ gives for each j, a q_j-dimensional quotient of $\mathbb{C}^n \otimes H^0(\mathcal{O}_{D_j})$, it is given by the composite map $H^0(\mathcal{O}^n \otimes \mathcal{O}_{D_j}) \to H^0(\mathfrak{F}_q \otimes \mathcal{O}_{D_j}) \to F_0^j(\mathfrak{F}_q)/F_1^j(\mathfrak{F}_q)$. Hence we get an embedding $\tilde{Q} \to Z = \mathrm{Grass}_{P(m)}(\mathbb{C}^n \otimes W) \times (\times_j \, \mathrm{Grass}_{qj}(\mathbb{C}^n \otimes H^0(\mathcal{O}_{D_j}))$. .... We denote a point of Z by $(P, (P_j))$ where $P\colon \mathbb{C}^n \otimes W \to U$, $P_j\colon \mathbb{C}^n \otimes H^0(\mathcal{O}_{D_j}) \to U_j$ are surjective maps, ...". Also in the expression for σ_H in Proposition 2·2 and in the proofs of Lemmas 2·5 and 2·6, one should replace "$P_j(H)$" by "$P_j(H \otimes H^0(\mathcal{O}_{D_j}))$".

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SUBJECT INDEX

AUTHOR INDEX